A Tetrahedron Approach for a Unique Closed-Form Solution of the Forward Kinematics of Six-dof Parallel Mechanisms with Multiconnected Joints

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This article presents a new formulation approach that uses tetrahedral geometry to determine a unique closed-form solution of the forward kinematics of six-dof parallel mechanisms with multiconnected joints. For six-dof parallel mechanisms that have been known to have eight solutions, the proposed formulation, called the *Tetrahedron Approach,* can find a unique closed-form solution of the forward kinematics using the three proposed Tetrahedron properties. While previous methods to solve the forward kinematics involve complicated algebraic manipulation of the matrix elements of the orientation of the moving platform, or closed-loop constraint equations between the moving and the base platforms, the Tetrahedron Approach piles up tetrahedrons sequentially to directly solve the forward kinematics. Hence, it allows significant abbreviation in the formulation and provides an easier systematic way of obtaining a unique closed-form solution.

1. INTRODUCTION

Six-dof Stewart platform–type parallel mechanisms consist of moving and base platforms and six links connected between the platforms. Such mechanisms have been applied in many robotic fields (such as flight simulators, haptic devices, micromechanisms and force-torque sensors) because they possess the especially attractive characteristics of a highly rigid structure, high local dexterity of orientation, and low sensitivity to external payload variation. The forward kinematics of six-dof parallel mechanisms determines the position and the orientation of the moving platform from the given link lengths. The forward kinematic solution is not generally expressed in an explicit-form equation, due to the fact that six links form several closed-chain loops in the connection between two platforms. There may exist multiple (from 8–24) solutions of the forward kinematics.[1–6,11] Many researches into the forward kinematics of six-dof parallel mechanisms focus on obtaining closed-form solutions from a single-variable polynomial equation.[5–14] In these works, a high-order polynomial

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equation has been reduced using geometrical simplification (by introducing extra sensors[5,7,8]) or using the symmetry and planarity of the moving and base platforms in the case of generalized six-dof parallel mechanisms.[9,10] In special cases of six-dof parallel mechanisms with multiconnected joints, it has been known that their forward kinematics can be easily formulated to be lower-order polynomial equations.[12–14] However, these previous works concerned with six-dof parallel mechanisms with multiconnected joints have concentrated on obtaining closed-form solutions of their forward kinematics, and proposed eight-order polynomial equations. In general, algebraic methods for solving a polynomial equation determine all possible solutions simultaneously. Checking for feasible solutions or determining an actual solution among all possible solutions may be a time-consuming and difficult task.

The purpose of this article is to present an effective formulation approach that determines a unique closed-form solution directly, without deriving a high-order polynomial equation. The proposed approach is applied to the forward kinematic analysis of several six-dof parallel mechanisms with multiconnected joints. The formulation procedure we introduce uses the geometric properties of a tetrahedron to solve the forward kinematics and is called the *Tetrahedron Approach*. The Tetrahedron Approach offers several advantages; unlike previous formulation approaches, the Tetrahedron Approach does not require geometric prerequisites (such as that the joint arrangements on the moving platform should all be symmetrically fixed in the same plane) and does not involve algebraic manipulation of orientation matrix elements or matrix manipulation of closed-loop vector equations in the formulation procedures.

This article is organized as follows. In Section 2, we introduce the Tetrahedron Proposition, the Tetrahedron Lemma, and the Tetrahedron Theorem, which are derived from the geometric properties of a tetrahedron and used in solving the forward kinematics of six-dof parallel mechanisms. Section 3 presents the formulation procedures to determine a unique closed-form solution of the forward kinematics for four six-dof parallel mechanisms with multiconnected joints. Section 4 explains how to find the forward kinematic solutions in a unique closed form when the four parallel mechanisms presented in Section 3 are modified. Finally, conclusions are presented.

2. TETRAHEDRON APPROACH TO SOLVING FORWARD KINEMATICS

In this section, we use the geometrical properties of a tetrahedron to present an efficient way, of solving forward kinematics. Three line constraints are sufficient to define a point in three-dimensional space, a geometric condition that can be represented by a tetrahedron as depicted in Figure 1(a). In the case of six-dof parallel mechanisms, a tetrahedron becomes a fundamental geometrical structure for determining the position of a particular point on the moving platform in 3D space. The Tetrahedron Approach simplifies the formulation procedure for solving the forward kinematics, such that it becomes a process of first identifying

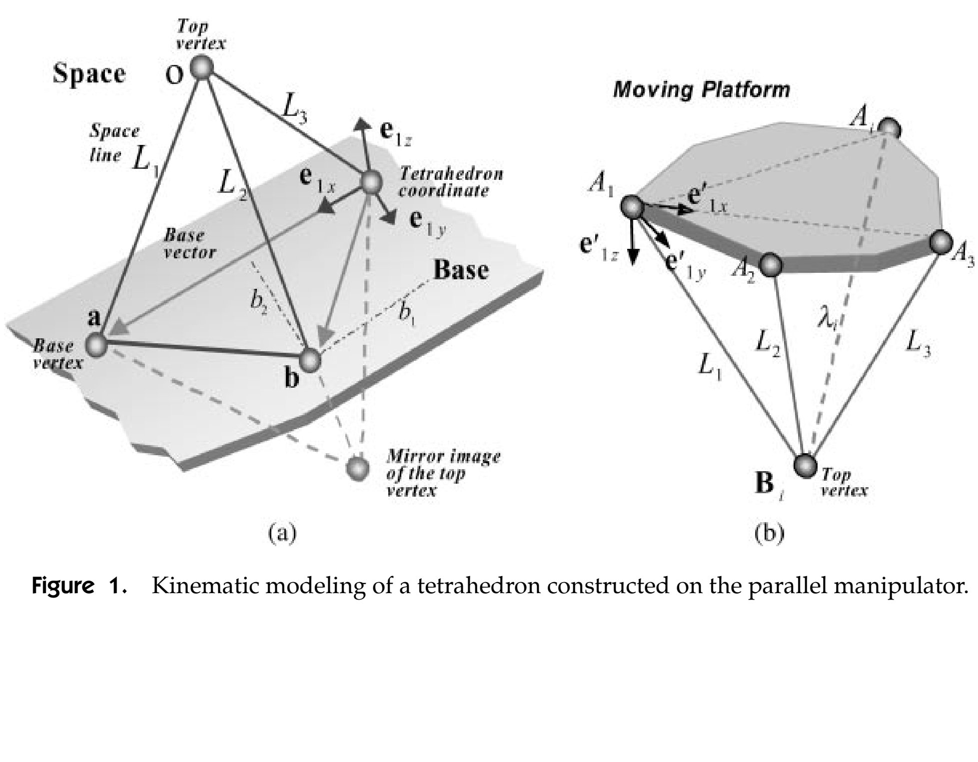


Figure 1. Kinematic modeling of a tetrahedron constructed on the parallel manipulator.

a tetrahedron based on the geometrical structures of given linkages, and then using it as a basis to identify and pile up another tetrahedron. These concepts are defined in the Tetrahedron Proposition, Tetrahedron Lemma, and Tetrahedron Theorem. The Tetrahedron Proposition uniquely identifies a tetrahedron based on the geometrical relationship between the base and moving platforms. The Tetrahedron Lemma is a method to identify another tetrahedron that satisfies the Tetrahedron Proposition. The Tetrahedron Theorem is a geometric condition that there exists a unique closed-form solution of the forward kinematics, and is used as a guideline to construct a formulation procedure to solve the forward kinematics using the Tetrahedron Proposition.

Notations used in this article for the forward kinematics of the six-dof parallel manipulators are defined as follows. Two main coordinates used to describe the kinematics of the parallel mechanism are: $[\mathbf{B}] = [\mathbf{X}, \mathbf{Y}, \mathbf{Z}]$, a base frame fixed on the base platform, and $[\mathbf{M}] = [\mathbf{u}, \mathbf{v}, \mathbf{w}]$, a moving frame fixed on the moving platform. O_A and O_B are fixed on the moving and the base platforms, respectively, and the position $\mathbf{H} = [X_C, Y_C, Z_C]^{\mathrm{T}}$ of the moving platform is a line vector connecting the two origins of the moving and the base frames. L_i is the ith link length. For the connecting joints, B_i indicates the ith joint on the base platform, and A_i indicates the ith joint on the moving platform. r and R are the radii of the moving and the base platforms, respectively. The moving frame $[\mathbf{M}]$ with respect to the base frame $[\mathbf{B}]$ is composed of the three column vectors of the orientation matrix with three Euler angles. The three mutual-orthogonal column vectors are denoted by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in order: $[\mathbf{R}] = [\mathbf{u}, \mathbf{v}, \mathbf{w}]$, and are called the orientation vectors. The moving frame $[\mathbf{M}]$ is identical to the orientation matrix $[\mathbf{R}]$.

Definition of Terms Used in the Tetrahedron Approach

Assume that there exist two known vectors among the six lines that compose a tetrahedron, as depicted in Figure 1(a). A plane defined by two vectors is called a *base*, which can be expressed with respect to a known reference coordinate. Three lines that lie on the base are called *base lines*. The three lines that connect the base to a vertex are called *space lines*. The vertex rising above the base from the three space lines is called a *top vertex* and the other three vertices are called *base vertices*. The vectors of the base lines and the space lines are called *base vectors* and *space vectors*, respectively. A set of three mutual-orthogonal unitary vectors formed from two base vectors is called a *tetrahedron coordinate*. A tetrahedron with two base vectors and three space lines is defined as a *directional tetrahedron,* which produces a direction from the base to the top vertex.

Tetrahedron Proposition *A top vertex can be uniquely determined with respect to a tetrahedron coordinate if there exist two known base vectors and three known space lines that geometrically compose a tetrahedron.*

Proof: Assume that two base vectors ($\mathbf{a}$, $\mathbf{b}$) and three space lines (L_1, L_2, L_3) are known for a tetrahedron, as shown in Figure 1(a). Then, a tetrahedron coordinate $[\mathbf{e}_{1x}, \mathbf{e}_{1y}, \mathbf{e}_{1z}]$ is formed with the two base vectors that are noncollinear:

$$\mathbf{e}_{1x} = \frac{\mathbf{a}}{\|\mathbf{a}\|}, \quad \mathbf{e}_{1y} = \frac{-(\mathbf{b}\cdot\mathbf{e}_{1x})\mathbf{e}_{1x} + \mathbf{b}}{\|-(\mathbf{b}\cdot\mathbf{e}_{1x})\mathbf{e}_{1x} + \mathbf{b}\|},$$
$$\mathbf{e}_{1z} = \pm\mathbf{e}_{1x} \times \mathbf{e}_{1y} \tag{1}$$

■

Thus, the space vector ($\mathbf{L}_3$) of a space line (L_3) can be obtained from triangular conditions among the three vertices of the tetrahedron, and represented with respect to the tetrahedron coordinate:

$$\mathbf{L}_3 = L_{3x}\mathbf{e}_{1x} + L_{3y}\mathbf{e}_{1y} + L_{3z}\mathbf{e}_{1z} \tag{2}$$

where

$$L_{3x} = \frac{L_3^2 + a^2 - L_1^2}{2a}, \quad L_{3y} = \frac{L_3^2 + b^2 - L_2^2 - 2L_{3x}(\mathbf{b}\cdot\mathbf{e}_{1x})}{2\mathbf{b}\cdot\mathbf{e}_{1y}},$$
$$L_{3z}^2 = L_3^2 - L_{3x}^2 - L_{3y}^2$$

The dot-product signs between the components of $\mathbf{L}_3$ in Eq. (2) are determined to be positive or negative according to the angles between two adjacent vectors:

$$\mathbf{a}\cdot\mathbf{L}_3 = \begin{cases} +a\,L_{3x}, & \text{if } a^2 + L_3^2 \geq L_1^2 \\ -a\,L_{3x}, & \text{else } a^2 + L_3^2 < L_1^2 \end{cases} \tag{3a}$$

$$\mathbf{b}\cdot\mathbf{L}_3 = \begin{cases} +[L_{3x}(\mathbf{b}\cdot\mathbf{e}_{1x}) + L_{3y}(\mathbf{b}\cdot\mathbf{e}_{1y})], & \text{if } b^2 + L_3^2 \geq L_2^2 \\ -[L_{3x}(\mathbf{b}\cdot\mathbf{e}_{1x}) + L_{3y}(\mathbf{b}\cdot\mathbf{e}_{1y})], & \text{else } b^2 + L_3^2 < L_2^2 \end{cases} \tag{3b}$$

In Eq. (1), there are two possible directions of the space vector ($\mathbf{L}_3$) according to the sign choice of $\mathbf{e}_{1z}$. The two possible directions are a binary choice ($\pm$). However, even though a top vertex is freely moving in the three-dimensional space, it is impossible in practice for the top vertex to be continuously changed to its mirror image. This arises from the fact that once the top vertex stays on a plane (the base) that consists of two known base vectors ($\mathbf{a}$, $\mathbf{b}$), it brings about a configuration singularity such that the choice of directions may disappear; the three degrees of freedom of the top vertex are reduced to two. As a result, the top vertex cannot move through the base; in practice, one of the two possible directions (solutions) is always feasible, and the other is imaginary. Therefore, when four vertices satisfy the Tetrahedron Proposition, the top vertex is uniquely determined.

For example, when the top vertex lies on the moving platform, we can eliminate the mirror image of the top vertex (the negative direction of $\mathbf{e}_{1z}$) and thus choose a positive direction for $\mathbf{e}_{1z}$. As a result, since there exist two base vectors ($\mathbf{a}$, $\mathbf{b}$) and three space lines (L_1, L_2, L_3) for a tetrahedron, the top vertex ($\mathbf{O}$) can be uniquely determined with respect to a tetrahedron coordinate as follows:

$$\mathbf{O} = \mathbf{L}_3 = \mathbf{a} + \mathbf{L}_1 = \mathbf{b} + \mathbf{L}_2 \tag{4}$$

In the Tetrahedron Proposition, we select the binary choices sequentially in the process of piling up directional tetrahedrons to solve the forward kinematics, eliminating the other choice (a mirror image), and thus find a true solution. If there exist less than three space lines in constructing a subsequent directional tetrahedron, this deficiency can be supplemented with the following Tetrahedron Lemma.

Tetrahedron Lemma *If three points that are not collinear are connected to a top vertex in three-dimensional space, and the lengths between the three points and the top vertex are known, the distance between the top vertex and any other point on the plane (defined by the three points) can be determined.*

Proof: When local positions fixed on a moving platform are all known, and at least three points (A_1, A_2, A_3) among them are commonly connected to a top vertex ($\mathbf{B}_i$), as shown in Figure 1(b), the tetrahedron with the three points (A_1, A_2, A_3) and the top vertex ($\mathbf{B}_i$) satisfies the Tetrahedron Proposition. A temporary tetrahedron coordinate can be defined from two noncollinear vectors on the moving platform. Therefore, a space vector ($\mathbf{L}_1$) can be expressed with respect to the temporary tetrahedron coordinate $[\mathbf{e}'_{1x}, \mathbf{e}'_{1y}, \mathbf{e}'_{1z}]$:

$$\mathbf{L}'_1 = L'_{1x}\mathbf{e}'_{1x} + L'_{1y}\mathbf{e}'_{1y} + L'_{1z}\mathbf{e}'_{1z} \tag{5a}$$

$$\mathbf{A}_2 = A_{2x}\mathbf{e}'_{1x} + A_{2y}\mathbf{e}'_{1y}, \quad \mathbf{A}_3 = A_{3x}\mathbf{e}'_{1x},$$
$$\mathbf{A}_i = A_{ix}\mathbf{e}'_{1x} - A_{iy}\mathbf{e}'_{1y} \tag{5b}$$

where

$$L'_{1x} = \frac{L_1^2 + \|\mathbf{A}_3 - \mathbf{A}_1\|^2 - L_3^2}{2\|\mathbf{A}_3 - \mathbf{A}_1\|}$$

$$L'_{1y} = \frac{L_1^2 + \|\mathbf{A}_2 - \mathbf{A}_1\|^2 - L_2^2 - 2L'_{1x}[(\mathbf{A}_2 - \mathbf{A}_1)\cdot\mathbf{e}'_{1x}]}{2(\mathbf{A}_2 - \mathbf{A}_1)\cdot\mathbf{e}'_{1y}},$$

$$L'^2_{1z} = L_1^2 - L'^2_{1x} - L'^2_{1y}, \quad \mathbf{e}'_{1x} = \frac{\mathbf{A}_3 - \mathbf{A}_1}{\|\mathbf{A}_3 - \mathbf{A}_1\|},$$

$$\mathbf{e}'_{1y} = \frac{-[(\mathbf{A}_2 - \mathbf{A}_1)\cdot\mathbf{e}'_{1x}]\mathbf{e}'_{1x} + (\mathbf{A}_2 - \mathbf{A}_1)}{\|-[(\mathbf{A}_2 - \mathbf{A}_1)\cdot\mathbf{e}'_{1x}]\mathbf{e}'_{1x} + (\mathbf{A}_2 - \mathbf{A}_1)\|},$$

$$\mathbf{e}'_{1z} = +\mathbf{e}'_{1x} \times \mathbf{e}'_{1y}.$$

Here, the components of $\mathbf{A}_2$, $\mathbf{A}_3$, and $\mathbf{A}_i$ are known with respect to $[\mathbf{e}'_{1x}, \mathbf{e}'_{1y}, \mathbf{e}'_{1z}]$ from the given parameters of the moving platform.

As a result, when three points are connected to a top vertex in the space and their lengths are given, the distance (λ_i) between the top vertex ($\mathbf{B}_i$) and any other point ($\mathbf{A}_i$) on the same plane is determined from the following difference of two vectors:

$$\lambda_i = \|\mathbf{L}'_1 - \mathbf{A}_i\| \tag{6}$$

■

We propose a geometric condition, the Tetrahedron Theorem, that there exists a unique closed-form solution of the forward kinematics, and present a guideline for constructing a formulation procedure to solve the forward kinematics using the proposed Tetrahedron Proposition and Lemma.

Tetrahedron Theorem *A unique closed-form solution of the forward kinematics can be found, if and only if there exist two top vertices on the moving platform that satisfy the Tetrahedron Proposition and a space line between the moving and the base platforms that is not collinear to a line connecting the two vertices.*

Proof: It is apparent that the posture of the moving platform in three-dimensional space is exactly determined when a position vector and three orientation vectors of the moving platform are known. Assuming that two top vertices ($\mathbf{A}_1$, $\mathbf{A}_i$) are known (as shown in Figure 2), one of them can be considered as the position vector ($\mathbf{H}$) of the moving platform. When the length (L_k) connecting another joint position (A_k) on the moving platform to a point ($\mathbf{B}_k$) on the base platform is given, the two top vertices ($\mathbf{A}_1$, $\mathbf{A}_i$) and the length (L_k) yield a directional tetrahedron that satisfies the Tetrahedron Proposition, since three known space lines (L_k, p_{1k}, p_{ik}) are given as fixed parameters and two base vectors ($\mathbf{B}_k - \mathbf{A}_1, \mathbf{B}_k - \mathbf{A}_i$) are

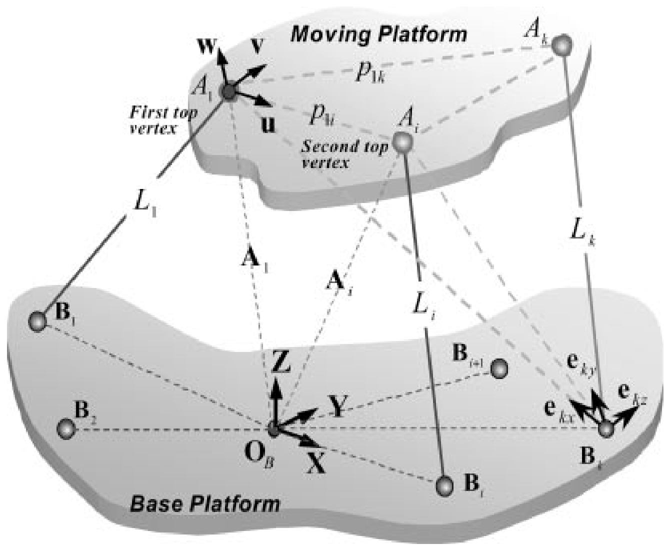


Figure 2. A parallel mechanism with two top vertices and another joint position on the moving platform with a known space line.

generated from the previous two top vertices. (p_{1i} and p_{1k} are the distances given between the two adjacent points on the moving platform.) It is worth noting that a rigid body in three-dimensional space is fixed by six constraints: $\mathbf{A}_1$ and $\mathbf{A}_i$ are taken to be three and two constraints, respectively, and $\mathbf{A}_k$ is taken to be the last constraint. This concept is equivalent to the Tetrahedron Theorem. ■

Thus the third top vertex ($\mathbf{A}_k$) is identified with respect to a tetrahedron coordinate $[\mathbf{e}_{kx}, \mathbf{e}_{ky}, \mathbf{e}_{kz}]$ that is formed from the two base vectors ($\mathbf{B}_k - \mathbf{A}_1$, $\mathbf{B}_k - \mathbf{A}_i$):

$$\mathbf{L}_k = L_{kx}\mathbf{e}_{kx} + L_{ky}\mathbf{e}_{ky} + L_{kz}\mathbf{e}_{kz}, \quad \mathbf{A}_k = \mathbf{L}_k + \mathbf{B}_k \tag{7}$$

where

$$L_{kx} = \frac{L_k^2 + \|\mathbf{A}_1 - \mathbf{B}_k\|^2 - \|\mathbf{A}_1 - \mathbf{A}_k\|^2}{2\|\mathbf{A}_1 - \mathbf{B}_k\|},$$

$$L_{ky} = \frac{L_k^2 + \|\mathbf{A}_i - \mathbf{B}_k\|^2 - \|\mathbf{A}_i - \mathbf{A}_k\|^2 - 2L_{kx}[(\mathbf{A}_i - \mathbf{B}_k)\cdot\mathbf{e}_{kx}]}{2(\mathbf{A}_i - \mathbf{B}_k)\cdot\mathbf{e}_{ky}},$$

$$L_{kz}^2 = L_k^2 - L_{kx}^2 - L_{ky}^2, \quad \mathbf{e}_{kx} = \frac{\mathbf{A}_1 - \mathbf{B}_k}{\|\mathbf{A}_1 - \mathbf{B}_k\|},$$

$$\mathbf{e}_{ky} = \frac{-[(\mathbf{A}_i - \mathbf{B}_k)\cdot\mathbf{e}_{kx}]\mathbf{e}_{kx} + (\mathbf{A}_i - \mathbf{B}_k)}{\|-[(\mathbf{A}_i - \mathbf{B}_k)\cdot\mathbf{e}_{kx}]\mathbf{e}_{kx} + (\mathbf{A}_i - \mathbf{B}_k)\|},$$

$$\mathbf{e}_{kz} = +\mathbf{e}_{kx} \times \mathbf{e}_{ky}$$

Let an orientation vector ($\mathbf{u}$) be collinear with the difference vector ($\mathbf{A}_i - \mathbf{A}_1$). As a result, the position ($\mathbf{H}$) can be selected as the first top vertex ($\mathbf{A}_1$) and the orientation vectors $[\mathbf{R}] = [\mathbf{u}, \mathbf{v}, \mathbf{w}]$ can be determined from the following geometric relationships between two line vectors that are obtained from the three top vertices on the moving platform:

$$\text{Position: } \mathbf{H} = \mathbf{A}_1 = [x, y, z]^T \tag{8a}$$

$$\text{Three orientation vectors: } \mathbf{u} = \frac{\mathbf{A}_i - \mathbf{A}_1}{p_{1i}},$$

$$\mathbf{w} = \mathbf{u} \times \frac{\mathbf{A}_k - \mathbf{A}_1}{p_{1k}}, \quad \mathbf{v} = \mathbf{w} \times \mathbf{u} \tag{8b}$$

where

$$\mathbf{R} = [\mathbf{u}, \mathbf{v}, \mathbf{w}] = \begin{bmatrix} C\alpha C\beta & C\alpha S\beta S\gamma - S\alpha C\gamma & C\alpha S\beta C\gamma + S\alpha S\gamma \\ S\alpha C\beta & S\alpha S\beta S\gamma + C\alpha C\gamma & S\alpha S\beta C\gamma - C\alpha S\gamma \\ -S\beta & C\beta S\gamma & C\beta C\gamma \end{bmatrix},$$

$$\beta = A\tan 2(S\beta, \sqrt{(C\alpha C\beta)^2 + (S\alpha C\beta)^2}),$$

$$\alpha = A\tan 2(S\alpha C\beta / C\beta, C\alpha C\beta / C\beta),$$

$$\gamma = A\tan 2(C\beta S\gamma / C\beta, C\beta C\gamma / C\beta),$$

$$C\alpha = \cos\alpha, \quad C\beta = \cos\beta, \quad C\gamma = \cos\gamma,$$

$$S\alpha = \sin\alpha, \quad S\beta = \sin\beta, \quad S\gamma = \sin\gamma$$

As a result, a unique closed-form solution (position: x, y, z, orientation: α, β, γ) of the forward kinematics of a six-dof parallel mechanism is determined from the geometric relationship between the two top vertices on the moving platform that satisfy the Tetrahedron Proposition and the space line connecting the two platforms. The two top vertices and the space line can easily be represented from the unique closed-form solution in Eqs. (8a) and (8b):

$$\text{The first top vertex: } \mathbf{A}_1 = \mathbf{H} \quad (9a)$$

$$\text{The second top vertex: } \mathbf{A}_i = \mathbf{A}_1 + p_{1i}\mathbf{u} \quad (9b)$$

$$\text{The space line: } L_k = \|\mathbf{A}_k - \mathbf{B}_k\| \quad (9c)$$

Thus the Tetrahedron Theorem is a necessary and sufficient condition that the forward kinematics of a parallel mechanism is determined in the form of a closed solution. Moreover, when a parallel mechanism has a geometric structure that satisfies the Tetrahedron Theorem, the position vector of the moving platform can be decomposed from the orientation vectors, since one among the three top vertices is equivalent to the position [**H**] and the geometric relationship between the three top vertices can yield the orientation vectors $[\mathbf{R}] = [\mathbf{u}, \mathbf{v}, \mathbf{w}]$.

3. UNIQUE CLOSED-FORM SOLUTION OF THE FORWARD KINEMATICS

Previous research has shown that six-dof parallel mechanisms with multiconnected joints as considered in this article have eight configurations,[1,12–14] as illustrated in Figures 3–6. Three Tetrahedron properties can simplify the complicated formulation procedures of the forward kinematics and reduce their eight solutions to a unique closed-form solution.

A tetrahedral geometry has been employed to solve the forward kinematics by Bruyninckx[12–14] and Han et al.[8] Bruyninckx used a tetrahedral geometry to analyze the forward kinematics of some parallel mechanisms with a tetrahedral structure, and focused on deriving a single-variable polynomial equation for obtaining a closed-form solution. Han et al. also used a tetrahedral geometry and extra sensor information to derive a closed-form solution of the forward kinematics. In Bruyninckx's work, the forward kinematics is formulated to obtain an eight-order polynomial equation through complicated formulation procedures. The eight-order polynomial equation leads to eight solutions. Han et al.'s work requires extra sensor information to determine a connecting joint position on the moving platform. Unlike these previous works, the Tetrahedron Approach uses the three Tetrahedron properties to pile up sequential tetrahedrons from which a closed-form solution of the forward kinematics is obtained.

The effectiveness of the proposed Tetrahedron Approach will be demonstrated in the formulation procedure to determine the forward kinematics of four six-dof parallel mechanisms, as presented in the following four cases.

Case 1. Forward Kinematics of a 3-2-1 Parallel Mechanism

A 3-2-1 parallel mechanism (as shown in Figure 3) has been shown to have eight solutions of the forward

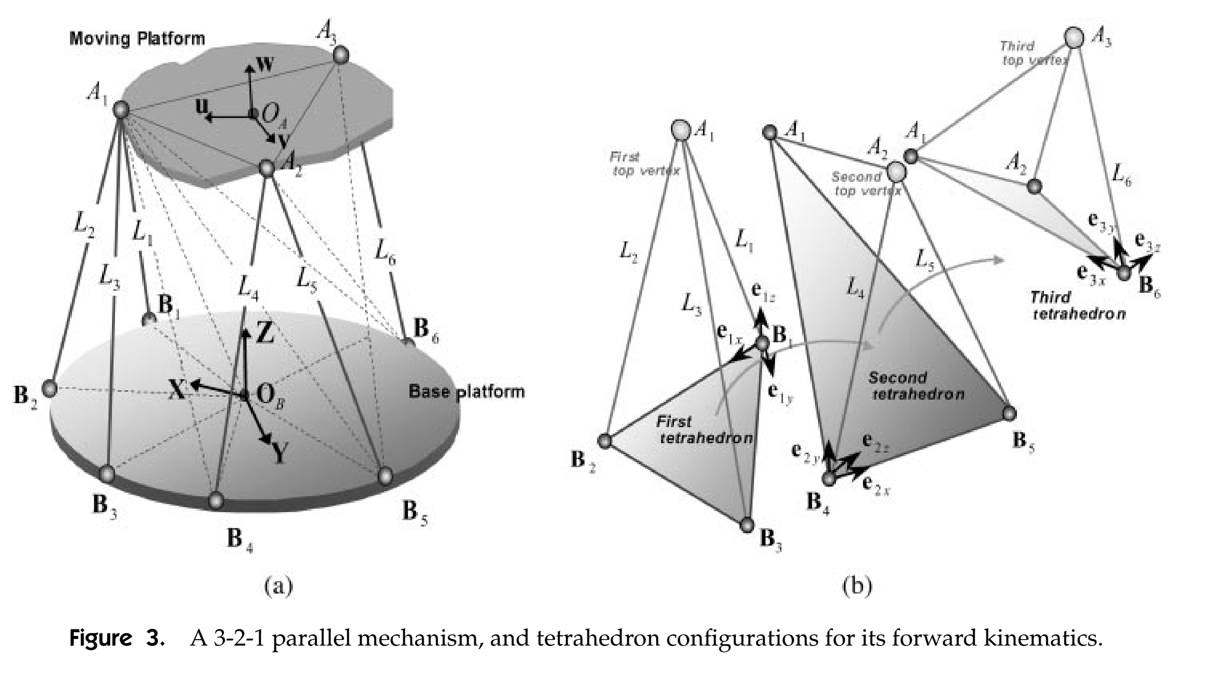


Figure 3. A 3-2-1 parallel mechanism, and tetrahedron configurations for its forward kinematics.

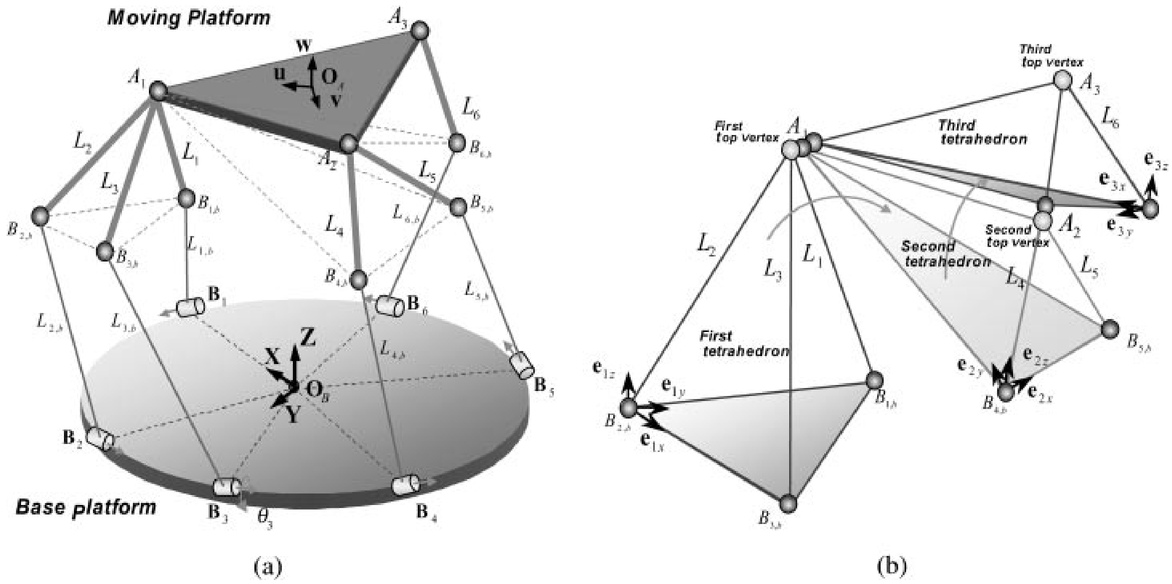


Figure 4. A 3-2-1 HEXA mechanism, and tetrahedron configurations for its forward kinematics.

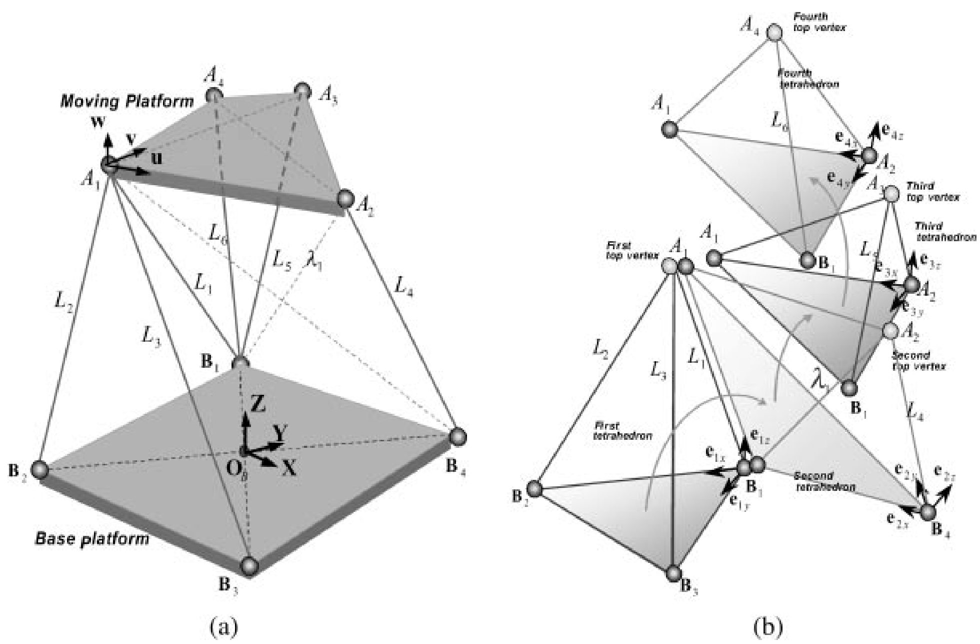


Figure 5. A $(3\text{-}1\text{-}1\text{-}1)^2$ parallel mechanism and tetrahedron configurations for its forward kinematics.

kinematics using Bezout's theorem.[11] Bruyninckx obtained four real solutions of the forward kinematics.[12] (The notation "3-2-1" denotes the number of linkages coinciding on the moving platform.) Four vertices $(A_1, \mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)$ yield the first directional tetrahedron, which has three known space lines (L_1, L_2, L_3) and two known base vectors $(\mathbf{B}_2 - \mathbf{B}_1, \mathbf{B}_3 - \mathbf{B}_1)$. By applying the Tetrahedron Proposition to the first directional tetrahedron, a space vector $(\mathbf{L}_1)$ and a top vertex $(\mathbf{A}_1)$ are obtained with respect to the first tetrahedron coordinate $[\mathbf{e}_{1x}, \mathbf{e}_{1y}, \mathbf{e}_{1z}]$, as follows:

$$\mathbf{L}_1 = L_{1x}\mathbf{e}_{1x} + L_{1y}\mathbf{e}_{1y} + L_{1z}\mathbf{e}_{1z}, \quad \mathbf{A}_1 = \mathbf{L}_1 + \mathbf{B}_1 \quad (10)$$

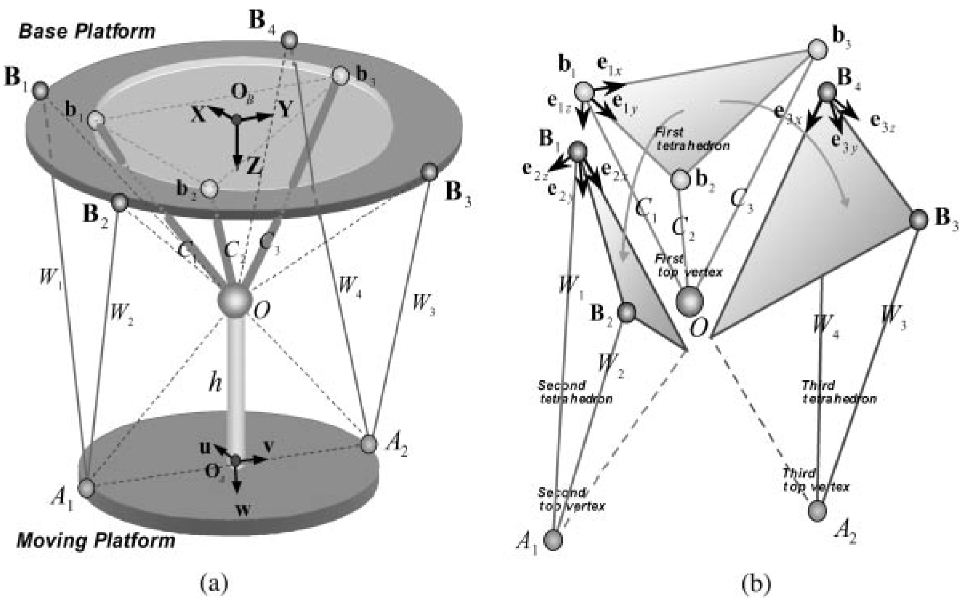


Figure 6. A 2-2 parallel mechanism with three cylinders and four wires, and tetrahedron configurations for its forward kinematics.

where

$$L_{1x} = \frac{L_1^2 + \|(\mathbf{B}_2 - \mathbf{B}_1)\|^2 - L_2^2}{2\|(\mathbf{B}_2 - \mathbf{B}_1)\|},$$

$$L_{1y} = \frac{L_1^2 + \|(\mathbf{B}_3 - \mathbf{B}_1)\|^2 - L_3^2 - 2L_{1x}[(\mathbf{B}_3 - \mathbf{B}_1)\cdot \mathbf{e}_{1x}]}{2(\mathbf{B}_3 - \mathbf{B}_1)\cdot \mathbf{e}_{1y}},$$

$$L_{1z}^2 = L_1^2 - L_{1x}^2 - L_{1y}^2, \quad \mathbf{e}_{1x} = \frac{\mathbf{B}_2 - \mathbf{B}_1}{\|\mathbf{B}_2 - \mathbf{B}_1\|},$$

$$\mathbf{e}_{1y} = \frac{-[(\mathbf{B}_3 - \mathbf{B}_1)\cdot \mathbf{e}_{1x}]\mathbf{e}_{1x} + (\mathbf{B}_3 - \mathbf{B}_1)}{\|-[(\mathbf{B}_3 - \mathbf{B}_1)\cdot \mathbf{e}_{1x}]\mathbf{e}_{1x} + (\mathbf{B}_3 - \mathbf{B}_1)\|},$$

$$\mathbf{e}_{1z} = +\,\mathbf{e}_{1x} \times \mathbf{e}_{1y} = \mathbf{Z}$$

As shown in Figure 3(a), the top vertex ($\mathbf{A}_1$) initially lies on the moving platform above the base platform. Therefore, since the top vertex ($\mathbf{A}_1$) must always stay above the base of the first tetrahedron, we can select the positive direction for $\mathbf{e}_{1z}$ after eliminating its mirror image, as addressed in the proof of the Tetrahedron Proposition.

Next, four vertices (A_1, $\mathbf{B}_4$, $\mathbf{B}_5$, A_2) yield the second directional tetrahedron, which satisfies the Tetrahedron Proposition because it has three known space lines ($\overline{A_1A_2}$, L_4, L_5) and two known base vectors ($\mathbf{B}_5 - \mathbf{B}_4$, $\mathbf{A}_1 - \mathbf{B}_4$) generated from the first directional tetrahedron. Therefore, the two space vectors ($\mathbf{L}_4$, $\mathbf{L}_5$) and the top vertex ($\mathbf{A}_2$) can be obtained with respect to the second tetrahedron coordinate [$\mathbf{e}_{2x}$, $\mathbf{e}_{2y}$, $\mathbf{e}_{2z}$], as follows:

$$\mathbf{L}_4 = L_{4x}\mathbf{e}_{2x} + L_{4y}\mathbf{e}_{2y} + L_{4z}\mathbf{e}_{2z}, \quad \mathbf{A}_2 = \mathbf{L}_4 + \mathbf{B}_4 \quad (11)$$

where

$$\mathbf{e}_{2x} = \frac{\mathbf{B}_5 - \mathbf{B}_4}{\|\mathbf{B}_5 - \mathbf{B}_4\|},$$

$$\mathbf{e}_{2y} = \frac{-[(\mathbf{A}_1 - \mathbf{B}_4)\cdot \mathbf{e}_{2x}]\mathbf{e}_{2x} + (\mathbf{A}_1 - \mathbf{B}_4)}{\|-[(\mathbf{A}_1 - \mathbf{B}_4)\cdot \mathbf{e}_{2x}]\mathbf{e}_{2x} + (\mathbf{A}_1 - \mathbf{B}_4)\|},$$

$$\mathbf{e}_{2z} = +\mathbf{e}_{2x} \times \mathbf{e}_{2y}$$

Though there exist two possible solutions of the top vertex ($\mathbf{A}_2$) according to the sign choice of $\mathbf{e}_{2z}$, only the positive direction of $\mathbf{e}_{2z}$ is also feasible, as for the first directional tetrahedron.

In a similar manner, four vertices (A_2, $\mathbf{B}_5$, $\mathbf{B}_6$, A_3) yield the third directional tetrahedron, which has three known space lines ($\overline{A_1A_3}$, $\overline{A_2A_3}$, L_6) and two known base vectors ($\mathbf{A}_1 - \mathbf{B}_6$, $\mathbf{A}_2 - \mathbf{B}_6$) generated by piling up the first and the second directional tetrahedrons. Since the third directional tetrahedron also satisfies the Tetrahedron Proposition, the top vertex ($\mathbf{A}_3$) can be obtained with respect to the third tetrahedron coordinate [$\mathbf{e}_{3x}$, $\mathbf{e}_{3y}$, $\mathbf{e}_{3z}$]:

$$\mathbf{L}_6 = L_{6x}\mathbf{e}_{3x} + L_{6y}\mathbf{e}_{3y} + L_{6z}\mathbf{e}_{3z}, \quad \mathbf{A}_3 = \mathbf{L}_6 + \mathbf{B}_6 \quad (12)$$

where

$$\mathbf{e}_{3x} = \frac{\mathbf{A}_1 - \mathbf{B}_6}{\|\mathbf{A}_1 - \mathbf{B}_6\|},$$
$$\mathbf{e}_{3y} = \frac{-[(\mathbf{A}_2 - \mathbf{B}_6)\cdot \mathbf{e}_{3x}]\mathbf{e}_{3x} + (\mathbf{A}_2 - \mathbf{B}_6)}{\|-[(\mathbf{A}_2 - \mathbf{B}_6)\cdot \mathbf{e}_{3x}]\mathbf{e}_{3x} + (\mathbf{A}_2 - \mathbf{B}_6)\|},$$
$$\mathbf{e}_{3z} = +\mathbf{e}_{3x} \times \mathbf{e}_{3y}$$

As a result, since the three top vertices ($\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$) obtained from three tetrahedrons lie on the moving platform, the position and orientation vectors of the moving platform can be determined by the following geometric relationships of the three top vertices on the moving platform in the case that they form a regular triangle, as shown in Figure 3:

$$\text{Position: } \mathbf{H} = \frac{\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3}{3} \quad (13a)$$

$$\text{Orientation vectors: } \mathbf{u} = \frac{\mathbf{A}_1 - \mathbf{H}}{r}, \quad \mathbf{v} = \frac{\mathbf{A}_2 - \mathbf{A}_3}{\sqrt{3}r}, \quad \mathbf{w} = \mathbf{u} \times \mathbf{v} \quad (13b)$$

It is worth remarking that three top vertices with each binary choice indicate the existence of eight possible solutions of the forward kinematics. However, since three top vertices on the moving platform are sequentially selected by applying the Tetrahedron Proposition, as depicted in the tetrahedron configurations of Figure 3, the forward kinematic solution is determined in the explicit form (unique closed-form) using the geometric relationship. We consider more complex parallel mechanisms in the following cases.

Case 2. Forward Kinematics of a 3-2-1 HEXA Mechanism

A 3-2-1 HEXA mechanism (as shown in Figure 4) is a six-dof fully parallel mechanism with R-RR-RRR legs and a 3-2-1 moving platform.[13] Like Case 1, the 3-2-1 HEXA mechanism has a total of eight solutions of the forward kinematics. While its overall structure is similar to the mechanism in Case 1, the 3-2-1 HEXA mechanism has six pin joints on the base platform.

As shown in Figure 4, since the position vectors of the six lower-link lengths ($L_{i,b}$) are obtained from angles (θ_i) measured from the rotary sensors mounted at the six pin joints on $\mathbf{B}_i$, their six position vectors ($\mathbf{B}_{i,b}$) can be determined. Therefore, four vertices ($A_1, \mathbf{B}_{1,b}, \mathbf{B}_{2,b}, \mathbf{B}_{3,b}$) yield the first directional tetrahedron with respect to the first tetrahedron coordinate [$\mathbf{e}_{1x}, \mathbf{e}_{1y}, \mathbf{e}_{1z}$], which has three known space lines (L_1, L_2, L_3) and two known base vectors ($\mathbf{B}_{2,b} - \mathbf{B}_{1,b}, \mathbf{B}_{3,b} - \mathbf{B}_{1,b}$) given by rotating the six fixed lengths ($\mathbf{L}_{i,b}$) with $\mathbf{R}_\mathbf{P}(\theta_i)$:

$$\mathbf{B}_{i,b} = [\mathbf{R}_\mathbf{P}(\theta_i)]\mathbf{L}_{i,b} \quad (14)$$

where

$$\mathbf{L}_{i,b} = [0, 0, L_{i,b}]^T, \quad \mathbf{R}_\mathbf{p}(\theta_i) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_i) & -\sin(\theta_i) \\ 0 & \sin(\theta_i) & \cos(\theta_i) \end{bmatrix}$$

$\mathbf{R}_\mathbf{P}(\theta_i)$ is a rotation matrix about a rotating angle (θ_i) at the ith pin joint ($\mathbf{B}_i$).

Consequently, three space vectors ($\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3$) and the top vertex ($\mathbf{A}_1$) are obtained with respect to the first tetrahedron coordinate:

$$\mathbf{L}_1 = L_{1x}\mathbf{e}_{1x} + L_{1y}\mathbf{e}_{1y} + L_{1z}\mathbf{e}_{1z}, \quad \mathbf{A}_1 = \mathbf{L}_1 + \mathbf{B}_{1,b} \quad (15)$$

where

$$\mathbf{e}_{1x} = \frac{\mathbf{B}_{2,b} - \mathbf{B}_{1,b}}{\|\mathbf{B}_{2,b} - \mathbf{B}_{1,b}\|},$$
$$\mathbf{e}_{1y} = \frac{[(\mathbf{B}_{3,b} - \mathbf{B}_{1,b})\cdot \mathbf{e}_{1x}]\mathbf{e}_{1x} + (\mathbf{B}_{3,b} - \mathbf{B}_{1,b})}{\|[(\mathbf{B}_{3,b} - \mathbf{B}_{1,b})\cdot \mathbf{e}_{1x}]\mathbf{e}_{1x} + (\mathbf{B}_{3,b} - \mathbf{B}_{1,b})\|},$$
$$\mathbf{e}_{1z} = +\mathbf{e}_{1x} \times \mathbf{e}_{1y}$$

As shown in Eq. (10), three components of $\mathbf{L}_1$ are all constant and determined by the three space lines (L_1, L_2, L_3) and the two base vectors ($\mathbf{B}_{2,b} - \mathbf{B}_{1,b}$, $\mathbf{B}_{3,b} - \mathbf{B}_{1,b}$). The top vertex ($\mathbf{A}_1$) is uniquely determined by the Tetrahedron Proposition.

After the formulation procedures of Eqs. (11) and (12) are applied here, we find that four vertices ($A_1, \mathbf{B}_{4,b}, \mathbf{B}_{5,b}, A_2$) yield the second directional tetrahedron, which also satisfies the Tetrahedron Proposition. Therefore, two space vectors ($\mathbf{L}_4, \mathbf{L}_5$) and the second top vertex ($\mathbf{A}_2$) can be obtained with respect to the second tetrahedron coordinate [$\mathbf{e}_{2x}, \mathbf{e}_{2y}, \mathbf{e}_{2z}$]. Four vertices ($A_1, \mathbf{B}_{6,b}, A_2, A_3$) yield the third directional tetrahedron and thus the top vertex ($\mathbf{A}_3$) can be obtained with respect to the third tetrahedron coordinate [$\mathbf{e}_{3x}, \mathbf{e}_{3y}, \mathbf{e}_{3z}$], as described in the tetrahedron configurations depicted in Figure 4. Since the three top vertices ($\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$) satisfy the Tetrahedron Theorem, the forward kinematics is determined in the explicit-form solution as in Eqs. (13a) and (13b).

Case 3. Forward Kinematics of a (3-1-1-1)^2 Fully Parallel Mechanism

A (3-1-1-1)^2 fully parallel mechanism is composed of two tetrahedral structures and has a maximum of eight solutions.[14] In this mechanism, three links (L_1, L_2, L_3) coincide at a joint (A_1) on the moving platform, and three links (L_1, L_5, L_6) coincide at a joint ($\mathbf{B}_1$) on the base platform, as shown in Figure 5. Though the three reverse links complicate the forward kinematic formulation, the forward kinematic solution can be uniquely determined by applying the Tetrahedron Lemma proposed in Section 2.

Four vertices ($A_1, \mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$) yield the first directional tetrahedron, which satisfies the Tetrahedron Proposition. The three space vectors ($\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3$) and the top vertex ($\mathbf{A}_1$) can be also obtained with respect to the first tetrahedron coordinate, as described in Eq. (10).

In the second directional tetrahedron with four vertices ($A_1, \mathbf{B}_1, A_2, \mathbf{B}_4$), there is an unknown space line (λ_1). Since four points (A_1, A_2, A_3, A_4) on the moving platform lie on the same plane and coincide at a point ($\mathbf{B}_1$), the length of the space line (λ_1) can be obtained by applying the Tetrahedron Lemma with respect to a temporary tetrahedron coordinate $[\mathbf{e}'_{1x}, \mathbf{e}'_{1y}, \mathbf{e}'_{1z}]$:

$$\mathbf{L}'_1 = L'_{1x}\mathbf{e}'_{1x} + L'_{1y}\mathbf{e}'_{1y} + L'_{1z}\mathbf{e}'_{1z}, \quad \mathbf{A}_2 = A_{2x}\mathbf{e}'_{1x} + A_{2y}\mathbf{e}'_{1y}, \lambda_1 = \|\mathbf{L}'_1 - \mathbf{A}_2\| \tag{16}$$

where

$$\mathbf{e}'_{1x} = \frac{\mathbf{A}_4 - \mathbf{A}_1}{\|\mathbf{A}_4 - \mathbf{A}_1\|}, \quad \mathbf{e}'_{1y} = \frac{-[(\mathbf{A}_3 - \mathbf{A}_1)\cdot\mathbf{e}'_{1x}]\mathbf{e}'_{1x} + (\mathbf{A}_3 - \mathbf{A}_1)}{\|-[(\mathbf{A}_3 - \mathbf{A}_1)\cdot\mathbf{e}'_{1x}]\mathbf{e}'_{1x} + (\mathbf{A}_3 - \mathbf{A}_1)\|}, \quad \mathbf{e}'_{1z} = +\mathbf{e}'_{1x} \times \mathbf{e}'_{1y}$$

Here, $\mathbf{A}_{2x}$ and $\mathbf{A}_{2y}$ are known from the given parameters of the moving platform. Three components ($L'_{1x}, L'_{1y}, L'_{1z}$) can be obtained as shown in Eq. (5).

Now, the four vertices ($A_1, \mathbf{B}_1, A_2, \mathbf{B}_4$) yield the second directional tetrahedron with three known space lines ($\lambda_1, L_4, \overline{A_1A_2}$) and two known base vectors ($\mathbf{B}_1 - \mathbf{B}_4, \mathbf{A}_1 - \mathbf{B}_4$), which satisfies the Tetrahedron Proposition. Two sets of four vertices of ($A_1, \mathbf{B}_1, A_2, A_3$) and ($A_1, \mathbf{B}_1, A_2, A_4$) yield the third and the fourth directional tetrahedrons, respectively, which also satisfy the Tetrahedron Proposition. From these results, the four top vertices ($\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$) are obtained through piling up four directional tetrahedrons, as shown in Figure 5. Since the four top vertices all lie on the moving platform, the position and the orientation vectors are uniquely determined in the form of a closed-form solution by:

$$\text{Position: } \mathbf{H} = \mathbf{A}_1 \tag{17a}$$

$$\text{Orientation vectors: } \mathbf{u} = \frac{\mathbf{A}_4 - \mathbf{A}_1}{\|\mathbf{A}_4 - \mathbf{A}_1\|}, \quad \mathbf{w} = \mathbf{u} \times \frac{\mathbf{A}_2 - \mathbf{A}_1}{\|\mathbf{A}_2 - \mathbf{A}_1\|}, \quad \mathbf{v} = \mathbf{w} \times \mathbf{u} \tag{17b}$$

Whereas Bruyninckx[14] performs complicated formulation procedures to obtain eight solutions, the Tetrahedron Approach piles up four tetrahedrons to directly obtain a unique closed-form solution.

Case 4. Forward Kinematics of a Hybrid Wire-Drive Parallel Mechanism with Three Cylinders and Four Wires

A hybrid wire-drive parallel mechanism (as shown in Figure 6) has been designed with mixture actuators of four high-power wire drives and three cylinder actuators, for heavy-duty manipulation in automated handling and assembly in construction.[15] The wire mechanism with three cylinder actuators requires at least four wires for six degrees of freedom. To the best of the authors' knowledge, the forward kinematics has not been reported in the literature. According to the Tetrahedron properties, it is expected that the hybrid parallel mechanism has eight possible configurations, since three directional tetrahedrons with each binary choice are employed to solve its forward kinematics.

Four vertices ($\mathbf{O}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$) yield the first directional tetrahedron, which has three known cylinder lengths (C_1, C_2, C_3) measured from the actuator sensors and two known base vectors ($\mathbf{b}_2 - \mathbf{b}_1$, $\mathbf{b}_3 - \mathbf{b}_1$). By applying the Tetrahedron Proposition to the first directional tetrahedron, the three space vectors ($\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$) and the top vertex ($\mathbf{O}$) can be uniquely obtained with respect to the first tetrahedron coordinate $[\mathbf{e}_{1x}, \mathbf{e}_{1y}, \mathbf{e}_{1z}]$.

The second and the third directional tetrahedrons, which can be derived with four vertices ($\mathbf{B}_1, \mathbf{B}_2, \mathbf{O}, A_1$) and ($\mathbf{B}_3, \mathbf{B}_4, \mathbf{O}, A_2$), respectively, satisfy the Tetrahedron Proposition. Therefore, three top vertices ($\mathbf{A}_1, \mathbf{A}_2, \mathbf{O}$) are obtained through forming three tetrahedrons, as shown in Figure 6. The position and orientation vectors of the moving platform are uniquely determined in the form of a closed-form solution using the geometric constraints of the moving platform:

$$\text{Orientation vectors: } \mathbf{u} = \frac{\mathbf{A}_2 - \mathbf{A}_1}{\|\mathbf{A}_2 - \mathbf{A}_1\|},$$

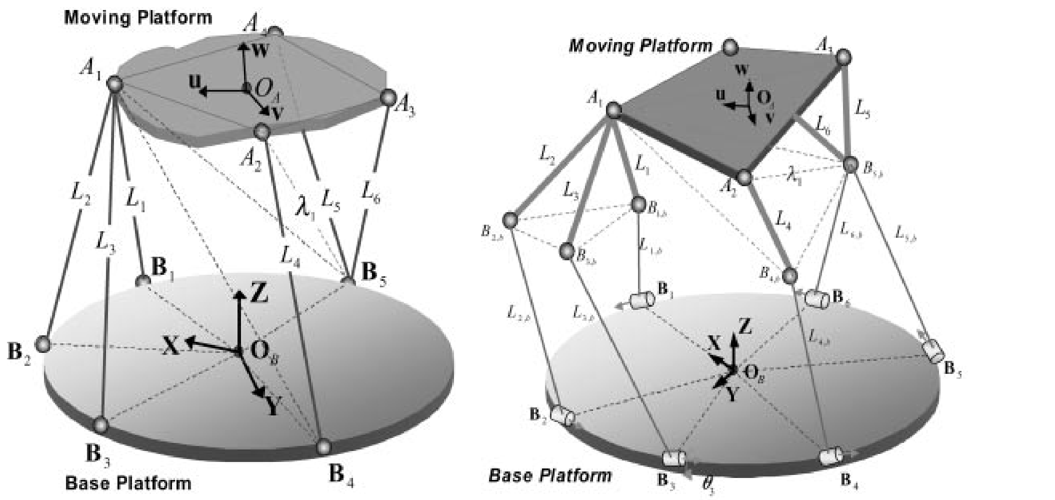
$$\mathbf{w} = \frac{\mathbf{A}_2 + \mathbf{A}_1 - 2\mathbf{O}}{\|\mathbf{A}_2 + \mathbf{A}_1 - 2\mathbf{O}\|}, \quad \mathbf{v} = \mathbf{u} \times \mathbf{w} \qquad (18a)$$

$$\text{Position: } \mathbf{H} = \mathbf{O} + h\mathbf{w} \qquad (18b)$$

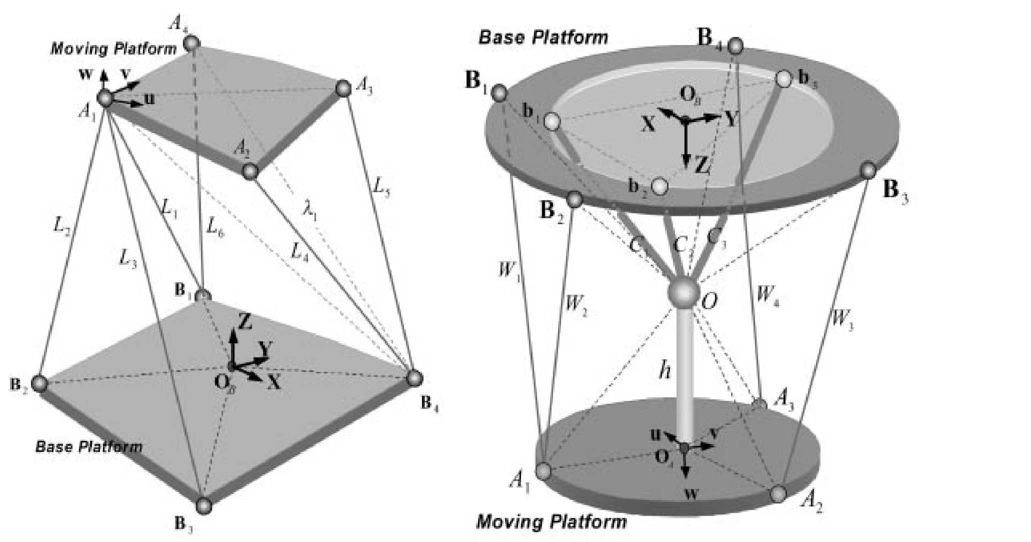
where h is a length between $\mathbf{O}$ and the origin of the moving platform.

4. APPLICATION OF THE TETRAHEDRON APPROACH TO MODIFIED SIX-DOF PARALLEL MECHANISMS

The Tetrahedron Approach using the geometric properties of a tetrahedron can also be directly applied to solve the forward kinematics of modified structures of the four six-dof parallel mechanisms addressed in Section 3. In Figures 7(a) and (b), the unknown length of the space line (λ_1) is determined by the Tetrahedron Lemma. Then, four directional tetrahedrons composed of four vertices (A_1, $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$), (A_1, A_2, $\mathbf{B}_4$, $\mathbf{B}_5$),



(a) A modified structure of the 3-2-1 parallel mechanism (b) A modified structure of the 3-2-1 HEXA mechanism



(c) A modified structure of the $(3\text{-}1\text{-}1\text{-}1)^2$ parallel mechanism (d) A modified structure of the 2-2 parallel mechanism

Figure 7. Four modified six-dof parallel mechanisms with unique closed-form solutions.

(A_1, A_2, $\mathbf{B}_5$, A_3), and (A_1, A_3, A_4, $\mathbf{B}_5$), all respectively satisfy the Tetrahedron Proposition, since their three space lines and two base vectors can all be obtained. Therefore, since the four top vertices ($\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$) on the moving platform satisfy the Tetrahedron Theorem, the position and the orientation vectors of the moving platform are uniquely determined in a closed-form solution.

The unknown length of a space line (λ_1) as shown in Figure 7(c) can also be determined by applying the Tetrahedron Lemma, because three space lines between three positions (A_1, A_2, A_3) and a vertex ($\mathbf{B}_4$) are known. Thus, four vertices (A_1, A_4, $\mathbf{B}_1$, $\mathbf{B}_4$) satisfy the Tetrahedron Proposition because their three space lines and two base vectors are known. The other tetrahedrons (A_1, A_3, $\mathbf{B}_1$, $\mathbf{B}_4$) and (A_1, A_2, A_3, $\mathbf{B}_4$) also satisfy the Tetrahedron Proposition, since their three space lines and two base vectors are known. In Figure 7(d), four directional tetrahedrons composed of four vertices ($\mathbf{O}$, $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$), ($\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{O}$, A_1), ($\mathbf{B}_3$, $\mathbf{B}_4$, $\mathbf{O}$, A_2), and ($\mathbf{B}_3$, $\mathbf{B}_4$, $\mathbf{O}$, A_3) all satisfy the Tetrahedron Proposition. Therefore, since the top vertices on the moving platform of the two parallel mechanisms of Figures 7(c) and (d) satisfy the Tetrahedron Theorem, the position and the orientation vectors can be uniquely determined in a closed-form solution.

As a result, the Tetrahedron Approach reduces the formulation procedures to just three or four calculations of tetrahedrons, without using extra sensors.

5. CONCLUSIONS

This article presents an effective formulation procedure, called the Tetrahedron Approach, that uses the geometric properties of a tetrahedron to obtain a unique closed-form solution of the forward kinematics of six-dof parallel mechanisms with multiconnected joints (which have been known to have eight configurations). Though the eight parallel mechanisms with multiconnected joints addressed in this article have different structures, it has been proved that we can use the three Tetrahedron properties to obtain a unique closed-form solution of their forward kinematics. The Tetrahedron Proposition uniquely determines a top vertex of a tetrahedron. The Tetrahedron Lemma determines the unknown length of a space line, to identify a next tetrahedron. The Tetrahedron Theorem determines a unique closed-form solution and simplifies the formulation procedure of the forward kinematics.

The Tetrahedron Approach allows for easy derivation and considerable simplification of the formulation procedure involved in determining the forward kinematic solutions for six-dof parallel mechanisms, and offers an analytic method to intuitively predict the existence of a unique closed-form solution, and the number of possible configurations.

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