Mobility of Overconstrained Parallel Mechanisms

Jian S. Dai
e-mail: Jian.Dai@KCL.ac.uk
Department of Mechanical Engineering,
School of Physical Sciences and Engineering,
King's College London,
University of London,
Strand, London, WC2R 2LS, United Kingdom

Zhen Huang
e-mail: huangz@ysu.edu.cn
Department of Mechanical Engineering,
Yanshan University,
PR China

Harvey Lipkin
e-mail: harvey.lipkin@me.gatech.edu
Woodruff School of Mechanical Engineering,
Georgia Institute of Technology,
Atlanta, GA 30332-0405

The Kutzbach–Grübler mobility criterion calculates the degrees of freedom of a general mechanism. However, the criterion can break down for mechanisms with special geometries, and in particular, the class of so-called overconstrained parallel mechanisms. The problem is that the criterion treats all constraints as active, even redundant constraints, which do not affect the mechanism degrees of freedom. In this paper we reveal a number of screw systems of a parallel mechanism, explore their inter-relationship and develop an original theoretical framework to relate these screw systems to motion and constraints of a parallel mechanism to identify the platform constraints, mechanism constraints and redundant constraints. The screw system characteristics and relationships are investigated for physical properties and a new approach to mobility analysis is proposed based on decompositions of motion and constraint screw systems. New versions of the mobility criterion are thus presented to eliminate the redundant constraints and accurately predict the platform degrees of freedom. Several examples of overconstrained mechanisms from the literature illustrate the results. [DOI: 10.1115/1.1901708]

Keywords: mobility, overconstraint, parallel mechanisms, platform, screw systems, screw theory, set-theoretic, motion, constraint redundancy, freedom reciprocal

1 Introduction

The study of parallel mechanisms dates back to 1897 when Bricard [1] investigated spherical motion profiles. The earliest industrial applications of parallel mechanisms are Gough's [2] tire testing machine and Stewart's [3] motion simulator. In the 1980s, the kinematics and design of full mobility parallel mechanisms were investigated by Earl and Rooney [4], Hunt [5], Mohamed and Duffy [6], Fichter [7], Waldron, Raghavan and Roth [8], and Gosselin and Angeles [9]. In particular, Mohamed and Duffy [6] investigated a parallel mechanism with serial chains as legs. Hunt [5] investigated parallel mechanisms with less than six degrees of freedom. Extensive studies were made in the 1990s in stiffness [10,11], workspace [12], kinematics [13], and dynamics [14] of parallel mechanisms. Herve [15] approached the structure of mechanisms using group theory. Recently, strong interest has been shown in generally arranged parallel mechanisms [16–18] and in lower mobility parallel mechanisms [19–26].

A parallel mechanism has a complex structure in terms of its motion and constraint. It is most naturally described by screw algebra [27–29]. Screws were used in the 1980s and 1990s to represent legs of the Gough–Stewart platform. Mohamed and Duffy [6] were the first to apply screws to a 6-6*R* parallel mechanism. The method was then used by Agrawal [30] to study parallel mechanisms and further extended by Lee, Duffy and Keler [31] with line geometry in developing the Jacobian matrix of planar parallel mechanisms. Recently the method was extended to parallel mechanisms with lower mobilities by Huang and Li [32,33] in type synthesis of lower mobility parallel mechanisms; by Zhao, Dai and Huang [34,35] in studying constraint characteristics; by Kong and Gosselin [36,37] in obtaining the wrench system of a CRR leg as a two-system; by Fang and Tsai [38] in structure synthesis; by Joshi and Tsai [39] in evaluating constraint screws of a three-*UPU* parallel mechanism, and by Bonev, Zlatanov and Gosselin [40] in identifying singular configurations. Further, Liu, Lou, and Li [41] used joint screws in *se*(3) to form the leg's Jacobian.

Ebert-Uphoff, Lee and Lipkin [17] used reciprocal screws to define a characteristic tetrahedron for the identification of singularities. Notash [42] used the reciprocal screw associated with an actuated main-arm joint of a branch to investigate the joint actuation for uncertainty configurations. Dasgupata and Mruthyunyaja [43] examined redundant forces by assembling the force transformation matrix based on joint screws.

More recently, the use of reciprocal screws has been extended to the mobility study of parallel mechanisms by Li and Huang [44] who employed reciprocal screws to examine the mobility of generally arranged 3-5*R* parallel mechanisms. Naturally, mobility has a substantial relationship with screw systems of a mechanism. A typical case is reflected in the (overconstrained) Sarrus mechanism which has the order of 5. As indicated by Waldron [45] and Hunt [46], the Grübler-Kutzbach criterion needs to be integrated with the screw system order. The use of a reciprocal screw system to study mobility has further been demonstrated in [47–49].

Screw systems have a substantial effect on mechanism motion and constraint. The study of screw systems has its origins with Ball [27]. In the 1960s Hunt [28] classified screw systems based on their principal screws. Dimentberg [50] enumerated a number of reciprocal screw systems. Waldron [51] applied screw systems to the constraint analysis of mechanisms and proposed a special relationship between two screw systems from the geometry of contact. The system classification was then followed by Nayak [52] based on instantaneous invariants. A further classification of screw system based on projective geometry was proposed by Gibson and Hunt [53], with a comprehensive examination by Rico-Martinez and Duffy [54,55]. The inter-relationship of these screw system was then identified by an inter-relationship theory proposed by Dai and Rees Jones [56,57].

Davies and Primose [58] examined mobility using the join and intersection of motion screw subspaces but did not include the reciprocal constraint screw subspaces in their development. Davies [59,60] applied the mechanical analogy of Kirchoff's laws to analyzing mechanisms as mechanical networks. Mobilities and constraint redundancies are determined by the row echelon reduction of screw matrices using either motion screws or constraint screws.

In this paper we comprehensively examine all motion and constraint screw systems of parallel mechanisms and exploit the rela-

Contributed by Design Engineering Division of ASME for publication in the Journal of Mechanical Engineering. Manuscript received June 17, 2004; final manuscript received October 14, 2004. Assoc. Editor: Constantinos Mavroidis.

tively simple topology of the mechanisms, compared to general mechanical networks, in order to accurately predict the mobility of overconstrained parallel mechanisms. The methodology relies on a combination of motion and constraint screw systems to potentially yield a simpler and straightforward analysis. To this end four basic screw systems (and their generators) are identified. The platform constraint screws are decomposed into screws that constrain the mechanism as a whole (the mechanism constraint screws) and those that further constrain the platform (the complementary constraint screws). The latter are further decomposed into active constraints (a complementary constraint screw basis) and redundant constraints (the virtual constraint screws). Then the number of redundant constraints is accounted for in the modified mobility criterion.

2 Screw Systems, Basis sets, and Spanning Multisets

In this paper it is necessary to distinguish between screw systems, sets of screws, and multisets of screws. For reference, these are briefly presented.

A *screw* $\boldsymbol{\$}$ is an element in a six-dimensional linear vector space with defined transformation properties. *Ray coordinates* are used to express the components of a screw so the first three components are in the direction of the screw axis. For motion screws the first three coordinates represent a rotation about the axis, and for constraint screws the first three coordinates represent a force along the axis, e.g. Hunt [28].

Two screws are *reciprocal* when $\boldsymbol{\$}_1 \circ \boldsymbol{\$}_2 = 0$, which in coordinate form is equivalent to $\boldsymbol{\$}_1^T \Delta \boldsymbol{\$}_2 = 0$ where

$$\boldsymbol{\Delta} = \begin{bmatrix} 0 & \boldsymbol{I} \\ \boldsymbol{I} & 0 \end{bmatrix} \tag{1}$$

is partitioned into 3×3 blocks and $\boldsymbol{I}$ is the identity matrix [61].

A *screw system* $\boldsymbol{S}$ is a linear vector subspace. The dimension of $\boldsymbol{S}$ is written as $\dim(\boldsymbol{S})$. The screw system $\boldsymbol{S}^r$ that is *reciprocal* to $\boldsymbol{S}$ is defined by

$$\boldsymbol{S}^r \equiv \{\boldsymbol{\$}_1 | \boldsymbol{\$}_1 \circ \boldsymbol{\$}_2 = 0, \forall\, \boldsymbol{\$}_2 \in \boldsymbol{S}\} \tag{2}$$

The dimensions of a system and its reciprocal are related by

$$\dim(\boldsymbol{S}) + \dim(\boldsymbol{S}^r) = 6 \tag{3}$$

The *union* and *intersection* of screw systems are written as $\boldsymbol{S}_1 \vee \boldsymbol{S}_2$ and $\boldsymbol{S}_1 \wedge \boldsymbol{S}_2$ respectively. For reference, equivalent terms for subspace union are *linear sum* and *span* which are often written as $\boldsymbol{S}_1 + \boldsymbol{S}_2$. If $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are disjoint, $\boldsymbol{S}_1 \wedge \boldsymbol{S}_2 = \emptyset$, then the sum is called direct and is often written as $\boldsymbol{S}_1 \oplus \boldsymbol{S}_2$.

The dimension law for two subspaces is well known,

$$\dim(\boldsymbol{S}_1 \vee \boldsymbol{S}_2) = \dim(\boldsymbol{S}_1) + \dim(\boldsymbol{S}_2) - \dim(\boldsymbol{S}_1 \wedge \boldsymbol{S}_2) \tag{4}$$

Screw systems also follow DeMorgan's laws [62–64],

$$(\boldsymbol{S}_1 \vee \boldsymbol{S}_2 \cdots \vee \boldsymbol{S}_n)^r = (\boldsymbol{S}_1^r \wedge \boldsymbol{S}_2^r \cdots \wedge \boldsymbol{S}_n^r) \tag{5}$$

$$(\boldsymbol{S}_1 \wedge \boldsymbol{S}_2 \cdots \wedge \boldsymbol{S}_n)^r = (\boldsymbol{S}_1^r \vee \boldsymbol{S}_2^r \cdots \vee \boldsymbol{S}_n^r) \tag{6}$$

where the reciprocal fulfills the role of the annihilator; see Birkhoff and MacLane [64].

Curly braces $\{\cdot\}$ are used to indicate a *set* which, by definition, contains unique elements. (There are two conventions regarding repeated elements in sets. In the more recent and explicit one used in this paper, no elements are repeated. This avoids confusion with multisets that do have repeated elements. In the older and implicit one, repeated elements may be listed but the multiplicities are completely ignored so, for example, $\{1,1,2\} = \{1,2\}$.) A *basis set* $\{\mathbf{S}\}$ is a linearly independent set of screws that span $\mathbf{S}$. The number of basis screws is the cardinal number of the set and is written as $\text{card}\{\mathbf{S}\}$. This is also equal to the dimension of the spanned subspace, $\text{card}\{\mathbf{S}\} = \dim(\mathbf{S})$. Since bases are sets, the usual set operations apply. The union, intersection, and difference of sets are designated by $\cup$, $\cap$, and $-$, respectively.

It is important to note that the union and intersection of sets are very different than the union and intersection of vector subspaces; to emphasize this fact distinct operational symbols are used. For example, if two completely distinct bases span the same subspace, there is a full intersection of their respective subspaces but no intersection of their respective basis elements.

Angle brackets $\langle\cdot\rangle$ are used to indicate a multiset that may contain repeated elements [65]. A *spanning multiset* $\langle \boldsymbol{S} \rangle$ is a collection of screws that span the screw system $\boldsymbol{S}$. The number of screws in a multiset is its cardinal number and is written as $\text{card}\langle \boldsymbol{S} \rangle$. This is always greater than or equal to the dimension of the spanned subspace, $\text{card}\langle \boldsymbol{S} \rangle \geqslant \dim(\boldsymbol{S})$. In a spanning multiset the screws may be repeated or linearly dependent. Therefore a basis set is a special instance of a spanning multiset so $\text{card}\langle \boldsymbol{S} \rangle \geqslant \dim\{\boldsymbol{S}\}$.

The following operations are defined for multisets $\langle \boldsymbol{S}_1 \rangle$ and $\langle \boldsymbol{S}_2 \rangle$: if $\boldsymbol{\$}$ occurs exactly n_1 times in $\langle \boldsymbol{S}_1 \rangle$ and exactly n_2 times in $\langle \boldsymbol{S}_2 \rangle$ then it occurs exactly (*i*) $n_1 + n_2$ times in $\langle \boldsymbol{S}_1 \rangle + \langle \boldsymbol{S}_2 \rangle$, (*ii*) $\max(n_1, n_2)$ times in $\langle \boldsymbol{S}_1 \rangle \cup \langle \boldsymbol{S}_2 \rangle$, (*iii*) $\min(n_1, n_2)$ times in $\langle \boldsymbol{S}_1 \rangle \cap \langle \boldsymbol{S}_2 \rangle$, and (*iv*) $\max(n_1 - n_2, 0)$ times in $\langle \boldsymbol{S}_1 \rangle - \langle \boldsymbol{S}_2 \rangle$. Multiset addition + simply combines all elements from both multisets into a single one, such as

$$\langle 2,2,3 \rangle + \langle 1,2,3,3 \rangle = \langle 1,2,2,2,3,3,3 \rangle \tag{7}$$

When no elements repeat then multisets are also ordinary sets and $\cup$, $\cap$, and $-$ take on the usual set meanings.

3 Four Basic Screw Systems of a Parallel Mechanism

A parallel mechanism consists of a moving platform (generalized coupler), a base (grounded link), and a number of kinematic branches (limbs) that connect the moving platform to the base. A kinematic branch is a serially connected chain of links and joints modeled by screws. For simplicity, it is assumed that the screws of each branch are linearly independent so redundancies and singular configurations within each branch are precluded.

Definition 1a: The *i*th *branch motion-screw system* $\boldsymbol{S}_{bi}$ spans the base-to-platform motion of the *i*th branch.

Definition 1b: The *i*th *branch constraint-screw system* $\boldsymbol{S}_{bi}^r$ spans the base-to-platform constraint of the *i*th branch. It is reciprocal to the *i*th *branch motion-screw system*.

Unions and intersections of the branch systems yield the four basic screw systems as follows.

The first two screw systems describe the motion and constraint of the platform to the base due to the parallel connections of the branches. Any relative motion must be allowed by every branch and, dually, any relative constraint can be imposed by any branch.

Definition 2a: The *platform motion-screw system* is the intersection of all p branch motion-screw systems,

$$\boldsymbol{S}_f = \boldsymbol{S}_{b1} \wedge \boldsymbol{S}_{b2} \cdots \wedge \boldsymbol{S}_{bp} \quad (m \equiv \dim(\boldsymbol{S}_f)) \tag{8}$$

Definition 2b: The *platform constraint-screw system* is the union of all p branch constraint-screw systems,

$$\boldsymbol{S}^r = \boldsymbol{S}_{b1}^r \vee \boldsymbol{S}_{b2}^r \cdots \vee \boldsymbol{S}_{bp}^r \quad (\mu \equiv \dim(\boldsymbol{S}^r)) \tag{9}$$

The remaining two screw systems describe the motion and constraint of the mechanism as a whole.

Definition 3a: The *mechanism motion-screw system* is the union of all p branch motion-screw systems,

$$\boldsymbol{S}_m = \boldsymbol{S}_{b1} \vee \boldsymbol{S}_{b2} \cdots \vee \boldsymbol{S}_{bp} \quad (d \equiv \dim(\boldsymbol{S}_m)) \tag{10}$$

Definition 3b: The *mechanism constraint-screw system* is the intersection of all p branch constraint-screw systems,

$$\boldsymbol{S}^c = \boldsymbol{S}_{b1}^r \wedge \boldsymbol{S}_{b2}^r \cdots \wedge \boldsymbol{S}_{bp}^r \quad (\lambda \equiv \dim(\boldsymbol{S}^c)) \tag{11}$$

$\mathbf{S}_m$ spans the allowable relative motions between all links of the mechanism. Dually, $\mathbf{S}^c$ spans the constraints common to every link of the mechanism.

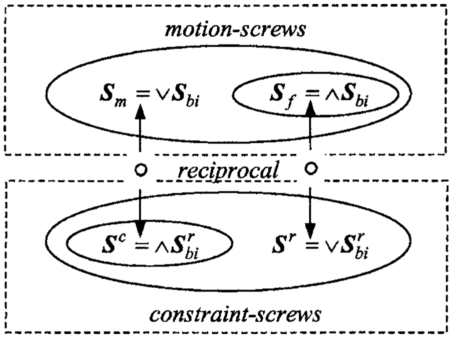

Fig. 1 The four basic screw subspaces in reciprocal pairs.

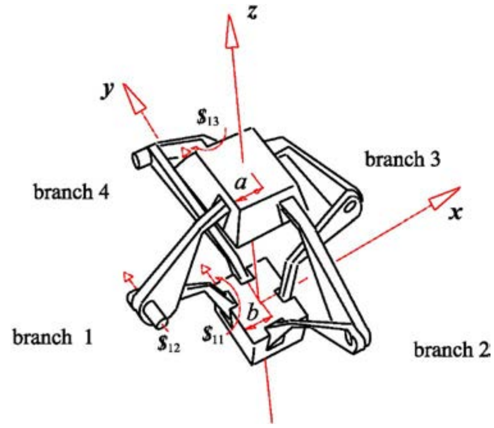

Fig. 2 4-*RRR* platform

The four systems form two dual pairs that are summarized in the following proposition and may be easily verified from DeMorgan's Laws (5) and (6) and the dimension relation of reciprocal screw systems (3).

Proposition 4: $\boldsymbol{S}_f$ and $\boldsymbol{S}^r$ form reciprocal screw systems for the motion and constraint of the platform; and $\boldsymbol{S}_m$ and $\boldsymbol{S}^c$ form reciprocal systems for the motion and constraint of all links of the mechanism,

$$(\boldsymbol{S}_f)^r = \boldsymbol{S}^r, \ \dim(\boldsymbol{S}_f) + \dim(\boldsymbol{S}^r) = m + \mu = 6 \tag{12}$$

$$(\boldsymbol{S}_m)^r = \boldsymbol{S}^c, \ \dim(\boldsymbol{S}_m) + \dim(\boldsymbol{S}^c) = d + \lambda = 6 \tag{13}$$

Since the intersection of screw systems is always contained in their union then it follows from Definitions 2 and 3 that:

Proposition 5: The platform motion-screw system is contained in the mechanism motion-screw system and, dually, the platform constraint-screw system contains the mechanism constraint-screw system,

$$\boldsymbol{S}_f \subseteq \boldsymbol{S}_m \tag{14}$$

$$\boldsymbol{S}^r \supseteq \boldsymbol{S}^c \tag{15}$$

The four subspaces are illustrated in Fig. 1 within their respective motion-screw or constraint-screw spaces along with their reciprocal correspondences.

Example 1: The 4-*RRR* platform in Fig. 2 is used to illustrate the basic screw systems. Each of the $p=4$ branches consists of three serially connected revolute joints with axes parallel to the base. The parallel mechanism is extracted from a ball mechanism [49] acting to connect two orthogonal circular loop chains in an octant. The platform and base are square hubs and the branches use scissors-like structures. The base is on the $x-y$ plane and the opposite branches are on the $x-z$ or $y-z$ planes. All joint axes are parallel to the x or y directions.

For branch 1 the joints are all parallel to the y axis. The branch motion-screw system is most easily described by its basis,

$$\{\boldsymbol{S}_{b1}\} = \left\{ \begin{array}{l} \$_{11} = [0 \ \ 1 \ \ 0 \ \ 0 \ \ 0 \ \ -b]^{\mathrm{T}}, \\ \$_{12} = [0 \ \ 1 \ \ 0 \ \ -c \ \ 0 \ \ -e]^{\mathrm{T}}, \\ \$_{13} = [0 \ \ 1 \ \ 0 \ \ -h \ \ 0 \ \ -a]^{\mathrm{T}} \end{array} \right\} \tag{16}$$

where a and b are the radii of the reference circles on the platform and base, respectively, h is the height of the platform, c is the distance between screw $\$_{12}$ and the x axis and e is the distance between screw $\$_{12}$ and the z axis. In the screw notation $\$_{ij}$, the first subscript i denotes the branch number and the second subscript j denotes the joint number within the branch. The remaining branches are similar,

$$\{\boldsymbol{S}_{b2}\} = \left\{ \begin{array}{l} \$_{21} = [1 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ b]^T, \\ \$_{22} = [1 \ \ 0 \ \ 0 \ \ 0 \ \ c \ \ e]^T, \\ \$_{23} = [1 \ \ 0 \ \ 0 \ \ 0 \ \ h \ \ a]^T \end{array} \right\} \tag{17}$$

$$\{\boldsymbol{S}_{b3}\} = \left\{ \begin{array}{l} \$_{31} = [0 \ \ 1 \ \ 0 \ \ 0 \ \ 0 \ \ b]^T, \\ \$_{32} = [0 \ \ 1 \ \ 0 \ \ -c \ \ 0 \ \ e]^T, \\ \$_{33} = [0 \ \ 1 \ \ 0 \ \ -h \ \ 0 \ \ a]^T \end{array} \right\} \tag{18}$$

$$\{\boldsymbol{S}_{b4}\} = \left\{ \begin{array}{l} \$_{41} = [1 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ -b]^T, \\ \$_{42} = [1 \ \ 0 \ \ 0 \ \ 0 \ \ c \ \ -e]^T, \\ \$_{43} = [1 \ \ 0 \ \ 0 \ \ 0 \ \ h \ \ -a]^T \end{array} \right\} \tag{19}$$

The mechanism motion-screw multiset combines the four basis sets,

$$\langle \boldsymbol{S}_m \rangle = \{\boldsymbol{S}_{b1}\} + \{\boldsymbol{S}_{b2}\} + \{\boldsymbol{S}_{b3}\} + \{\boldsymbol{S}_{b4}\} \tag{20}$$

where $\mathrm{card}\langle \boldsymbol{S}_m \rangle = 12$. However $\langle \boldsymbol{S}_m \rangle$ only contains five linearly independent screws so a nonunique basis for the subspace $\boldsymbol{S}_m$ can be selected as

$$\{\boldsymbol{S}_m\} = \left\{ \begin{array}{l} \$_{11} = [0 \ \ 1 \ \ 0 \ \ 0 \ \ 0 \ \ -b]^T, \\ \$_{22} = [1 \ \ 0 \ \ 0 \ \ 0 \ \ c \ \ e]^T, \\ \$_{32} = [0 \ \ 1 \ \ 0 \ \ -c \ \ 0 \ \ e]^T, \\ \$_{42} = [1 \ \ 0 \ \ 0 \ \ 0 \ \ c \ \ -e]^T, \\ \$_{43} = [1 \ \ 0 \ \ 0 \ \ 0 \ \ h \ \ -a]^T \end{array} \right\} \tag{21}$$

This shows that the relative motions of all links in the mechanism are contained within a 5-degree-of-freedom screw system.

A basis for each *branch constraint-screw system* is easily calculated,

$$\{\boldsymbol{S}^r_{b1}\} = \left\{ \begin{array}{l} \$^r_{11} = [0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 1]^T, \\ \$^r_{12} = [0 \ \ 0 \ \ 0 \ \ 1 \ \ 0 \ \ 0]^T, \\ \$^r_{13} = [0 \ \ 1 \ \ 0 \ \ 0 \ \ 0 \ \ 0]^T \end{array} \right\} \tag{22}$$

$$\{\boldsymbol{S}^r_{b2}\} = \left\{ \begin{array}{l} \$^r_{21} = [0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 1]^T, \\ \$^r_{22} = [0 \ \ 0 \ \ 0 \ \ 0 \ \ 1 \ \ 0]^T, \\ \$^r_{23} = [1 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0]^T \end{array} \right\} \tag{23}$$

$$\{\boldsymbol{S}^r_{b3}\} = \left\{ \begin{array}{l} \$^r_{31} = [0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 1]^T, \\ \$^r_{32} = [0 \ \ 0 \ \ 0 \ \ 1 \ \ 0 \ \ 0]^T, \\ \$^r_{33} = [0 \ \ 1 \ \ 0 \ \ 0 \ \ 0 \ \ 0]^T \end{array} \right\} \tag{24}$$

$$\{\boldsymbol{S}^r_{b4}\} = \left\{ \begin{array}{l} \$^r_{41} = [0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 1]^T, \\ \$^r_{42} = [0 \ \ 0 \ \ 0 \ \ 0 \ \ 1 \ \ 0]^T, \\ \$^r_{43} = [1 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0]^T \end{array} \right\} \tag{25}$$

Note that $\{\boldsymbol{S}^r_{b1}\} = \{\boldsymbol{S}^r_{b3}\}$ and $\{\boldsymbol{S}^r_{b2}\} = \{\boldsymbol{S}^r_{b4}\}$ so the constraint subspaces correspond, $\boldsymbol{S}^r_{b1} = \boldsymbol{S}^r_{b3}$ and $\boldsymbol{S}^r_{b2} = \boldsymbol{S}^r_{b4}$. (This also implies the dual relations for the motion subspaces, $\mathbf{S}_{b1}=\mathbf{S}_{b3}$ and $\mathbf{S}_{b2}=\mathbf{S}_{b4}$ even though the corresponding bases differ.)

The platform constraint-screw multiset combines the four basis sets,

$$\langle \mathbf{S}^r\rangle = \{\mathbf{S}^r_{b1}\} + \{\mathbf{S}^r_{b2}\} + \{\mathbf{S}^r_{b3}\} + \{\mathbf{S}^r_{b4}\} \tag{26}$$

where card$\langle \mathbf{S}^r\rangle=12$. However $\langle \mathbf{S}^r\rangle$ only contains five linearly independent screws so a nonunique basis for the subspace $\mathbf{S}^r$ can be selected as

$$\{\mathbf{S}^r\} = \begin{Bmatrix} \$^r_{11} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1]^T, \\ \$^r_{12} = [0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0]^T, \\ \$^r_{13} = [0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T, \\ \$^r_{22} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0]^T, \\ \$^r_{23} = [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T \end{Bmatrix} \tag{27}$$

This shows that the platform motion is subject to five independent constraints.

Taking the reciprocal of $\mathbf{S}^r$ gives the platform motion-screw system $\mathbf{S}_f$ with the basis

$$\{\mathbf{S}_f\} = \{\$_{f1} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1]^T\} \tag{28}$$

which is in the span of each of the branch motion-screw systems $\mathbf{S}_{bi}$. This shows that the platform has one degree of freedom which is a translation along the z axis.

Dually, taking the reciprocal of $\mathbf{S}_m$ gives the mechanism constraint-screw system $\mathbf{S}^c$ with the basis

$$\{\mathbf{S}^c\} = \{\$^c_1 = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1]^T\} \tag{29}$$

which is in the span of each branch of the constraint-screw system. This shows that every link of the mechanism is subject to this common constraint which is a couple about the z axis.

Example 2: A further example is given by a 3-*RR*(*RRR*) parallel mechanism proposed by Li and Huang [33], see Fig. 3. The mechanism has three similar branches, each with five revolute joints. For each branch, counting from the base: joints 1 and 2 are perpendicular to the base; joint 3 is parallel with the base plane; and the last two revolute joints are arranged so joint axes 3, 4, and 5 all intersect at a point. This arrangement is the same for the remaining two branches so the top nine joints of the mechanism all intersect at one common point. This point is movable on a plane which is parallel to the base. To further simplify the problem, the branches are positioned symmetrically so the coordinates of one can be transformed into the other two by successive 120 deg rotations about the z axis.

The coordinate origin is at the intersection point, the z axis is normal to the base, and the x axis is aligned with the third joint axis of branch 1. The screws of branch 1 are

$$\{\mathbf{S}_{b1}\} = \begin{Bmatrix} \$_{11} = [0 \;\; 0 \;\; 1 \;\; p_{11} \;\; q_{11} \;\; 0]^T, \\ \$_{12} = [0 \;\; 0 \;\; 1 \;\; 0 \;\; q_{12} \;\; 0]^T, \\ \$_{13} = [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T, \\ \$_{14} = [l_{14} \;\; m_{14} \;\; n_{14} \;\; 0 \;\; 0 \;\; 0]^T, \\ \$_{15} = [l_{15} \;\; m_{15} \;\; n_{15} \;\; 0 \;\; 0 \;\; 0]^T \end{Bmatrix} \tag{30}$$

The branch constraint-screw basis is

$$\{\mathbf{S}^r_{b1}\} = \{\$^r_{11} = [0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0]^T\} \tag{31}$$

This represents a constraint force on the platform along the z axis. Since the remaining two branches have the same kinematic structure, and the three branches are arranged symmetrically around the base circle of diameter b, a coordinate transformation about the z axis can be applied to branch 1 motion-screw systems. The constraint-screw bases for branches 2 and 3 are then

$$\{\mathbf{S}^r_{b2}\} = \{\$^r_{21} = [0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0]^T\}$$

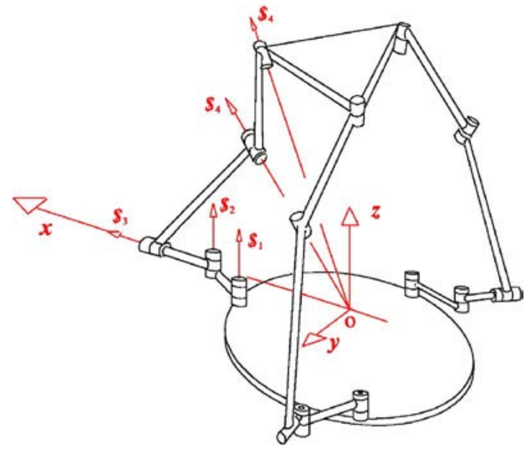

Fig. 3 3-*RR*(*RRR*) parallel mechanism

$$\{\mathbf{S}^r_{b3}\} = \{\$^r_{31} = [0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0]^T\} \tag{32}$$

which also follows more directly from the symmetry of the problem.

All three combine to form the platform constraint-screw multiset,

$$\langle \mathbf{S}^r\rangle = \langle \$^r_{11}, \$^r_{21}, \$^r_{31}\rangle \tag{33}$$

which, for this example, is also equal to the mechanism constraint-screw multiset $\langle \mathbf{S}^c\rangle$. Further $\mathbf{S}^r=\mathbf{S}^c$ and has dimension 1. It follows from Propositions 4 and 5 that the mechanism motion-screw system and the platform motion-screw system are equivalent for this example, $\mathbf{S}_f=\mathbf{S}_m$, and of dimension 5. A basis of motion screws is

$$\{\mathbf{S}_f\} = \begin{Bmatrix} \$_{f1} = [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T, \\ \$_{f2} = [0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T, \\ \$_{f3} = [0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0]^T, \\ \$_{f4} = [0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0]^T, \\ \$_{f5} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0]^T \end{Bmatrix} \tag{34}$$

This system has three rotational freedoms and two translational freedoms.

4 Complementary and Virtual Constraint Screws

Each reciprocal branch screw system can be expressed as the direct sum

$$\mathbf{S}^r_{bi} = \mathbf{S}^c \vee \mathbf{S}^r_{ci}, \quad (\mathbf{S}^c \wedge \mathbf{S}^r_{ci} = \varnothing), \tag{35}$$

where $\mathbf{S}^r_{ci}$ is called the ith *complementary* branch constraint-screw system. $\mathbf{S}^r_{bi}$ represents the total constraint on the platform by the ith branch which is decomposed into two parts. First $\mathbf{S}^c$ constrains the platform motion to remain within the mechanism motion system $\mathbf{S}_m$. Second, the remaining portion $\mathbf{S}^r_{ci}$ further constrains the platform motion to remain within branch motion system $\mathbf{S}_{bi}$ where $\mathbf{S}_{bi}\subseteq\mathbf{S}_m$. The corresponding basis elements also form two disjoint sets,

$$\{\mathbf{S}^r_{bi}\} = \{\mathbf{S}^c\} \cup \{\mathbf{S}^r_{ci}\}, \quad (\{\mathbf{S}^c\} \cap \{\mathbf{S}^r_{ci}\} = \varnothing) \tag{36}$$

Combining the branch constraint-screw bases using the multiset addition operator + yields a multiset that is decomposed into a pair of disjoint multisets,

$$\langle \mathbf{S}^r\rangle = \langle \mathbf{S}^c\rangle + \langle \mathbf{S}^r_c\rangle, \quad (\langle \mathbf{S}^c\rangle \cap \langle \mathbf{S}^r_c\rangle = \varnothing) \tag{37}$$

where,

$$\langle \boldsymbol{S}^r \rangle = \{\boldsymbol{S}^r_{b1}\} + \{\boldsymbol{S}^r_{b2}\} \cdots + \{\boldsymbol{S}^r_{bp}\}$$
$$\langle \boldsymbol{S}^c \rangle = \{\boldsymbol{S}^c\}$$
$$\{\boldsymbol{S}^c\} \cdots$$
$$\{\boldsymbol{S}^c\}, (p \text{ times})$$
$$\langle \boldsymbol{S}^r_c \rangle = \{\boldsymbol{S}^r_{c1}\}$$
$$\{\boldsymbol{S}^r_{c2}\} \cdots$$
$$\{\boldsymbol{S}^r_{cp}\}$$

For multiset addition the cardinalities are just summed,

$$\text{card}\langle \boldsymbol{S}^r \rangle = \text{card}\langle \boldsymbol{S}^c \rangle + \text{card}\langle \boldsymbol{S}^r_c \rangle = p \cdot \text{card}\{\boldsymbol{S}^c\} + \text{card}\langle \boldsymbol{S}^r_c \rangle \quad (38)$$

where,

$$\text{card}\langle \boldsymbol{S}^r \rangle = \sum \text{card}\{\boldsymbol{S}^r_{bi}\}$$

$$\text{card}\langle \boldsymbol{S}^c \rangle = p \cdot \text{card}\{\boldsymbol{S}^c\}$$

Additionally since the multisets are disjoint so are the bases and

$$\text{card}\{\boldsymbol{S}^r\} = \text{card}\{\boldsymbol{S}^c\} + \text{card}\{\boldsymbol{S}^r_c\} \quad (39)$$

$\langle \boldsymbol{S}^r \rangle$ represents the total constraint on the platform by all branches, which generally contains redundant constraints. $\langle \boldsymbol{S}^c \rangle$ represents the portion of the total constraint on the platform that restricts it to the mechanism motion system $\langle \boldsymbol{S}_m \rangle$. The constraint is $p-1$ times redundant. $\langle \boldsymbol{S}^r_c \rangle$ represents the portion of the total constraint that further restricts the platform motion to remain within the platform motion system $\langle \boldsymbol{S}_f \rangle$ where $\langle \boldsymbol{S}_f \rangle \subseteq \langle \boldsymbol{S}_m \rangle$.

Generally, $\langle \boldsymbol{S}^r_c \rangle$ contains redundant constraints which need to be identified. Continuing the decomposition of (37), the complementary constraint multiset is expressed as

$$\langle \boldsymbol{S}^r_c \rangle = \{\boldsymbol{S}^r_c\} + \langle \boldsymbol{S}^r_v \rangle \quad (40)$$

where $\{\boldsymbol{S}^r_c\}$ is a largest linearly independent set of screws in $\langle \boldsymbol{S}^r_c \rangle$ and the remaining screws make up the *virtual constraint multiset*, $\langle \boldsymbol{S}^r_v \rangle = \langle \boldsymbol{S}^r_c \rangle - \{\boldsymbol{S}^r_c\}$. Clearly the decomposition is not unique. The cardinality relation is

$$\text{card}\langle \boldsymbol{S}^r_c \rangle = \text{card}\{\boldsymbol{S}^r_c\} + \text{card}\langle \boldsymbol{S}^r_v \rangle \quad (41)$$

As above, $\langle \boldsymbol{S}^r_c \rangle$ represents the additional constraints that further limit the platform motion from $\boldsymbol{S}_m$ to $\boldsymbol{S}_f$. However, this can be done minimally by the linearly independent constraints $\{\boldsymbol{S}^r_c\}$. This means that the remaining constraints $\langle \boldsymbol{S}^r_v \rangle$ are redundant and not required to keep the platform motion within $\boldsymbol{S}_f$. This is summarized by the following.

Proposition 6: A virtual constraint-screw multiset $\langle \boldsymbol{S}^r_v \rangle$ is dependent on the complementary constraint-screw system $\{\boldsymbol{S}^r_c\}$. It forms redundant constraints [44,49] that do not affect the motion of the platform.

Combining the multiset relations (37) and (40) gives the fundamental platform constraint-screw decomposition. (For completeness it is noted that there is a corresponding decomposition for $\langle \boldsymbol{S}_m \rangle$ but its use in the context of this paper is limited).

$$\langle \boldsymbol{S}^r \rangle = \langle \boldsymbol{S}^c \rangle + \{\boldsymbol{S}^r_c\} + \langle \boldsymbol{S}^r_v \rangle \quad (42)$$

A corresponding cardinality relation is given by subtracting (39) from (38) and using (41),

$$\begin{aligned}\text{card}\langle \boldsymbol{S}^r \rangle - \text{card}\{\boldsymbol{S}^r\} &= (p-1) \cdot \text{card}\{\boldsymbol{S}^c\} + \text{card}\langle \boldsymbol{S}^r_c \rangle - \text{card}\{\boldsymbol{S}^r_c\} \\ &= (p-1) \cdot \text{card}\{\boldsymbol{S}^c\} + \text{card}\langle \boldsymbol{S}^r_v \rangle \quad (43)\end{aligned}$$

or more concisely as

$$c = (p-1)\lambda + v \quad (44)$$

where

$$c \equiv \text{card}\langle \boldsymbol{S}^r \rangle - \text{card}\{\boldsymbol{S}^r\}$$

$$v \equiv \text{card}\langle \boldsymbol{S}^r_v \rangle$$

$$\lambda \equiv \dim(\boldsymbol{S}^c) = \text{card}\{\boldsymbol{S}^c\} \quad (45)$$

and c represents the total number of redundant screws in the platform constraint-screw multiset. The first term in (44), $(p-1)\lambda$, counts the redundant mechanism constraint screws due to the joint motion-screws belonging to the d system $\boldsymbol{S}_m$ and is completely independent of how the joints are arranged. However, the platform is further constrained by the particular arrangement of the joints into branches. The second term, ν, counts the additional platform constraint redundancies introduced by the particular arrangement of joints in the mechanism.

Once the branch platform constraints $\boldsymbol{S}^r_{bi}$ are determined it is very simple to compute the number of redundant constraints c from Eq. (45). It is only necessary to count the screws in $\langle \boldsymbol{S}^r \rangle$ and then subtract the number of linear independent screws in $\langle \boldsymbol{S}^r \rangle$ which may be determined by Gauss elimination, for example. The expressions in (44) and (45) are used to augment the mobility criterion for overconstrained parallel mechanisms in the following section.

5 Modified Mobility Criterion

The generalized Kutzbach-Grübler [28,66–68] mobility criterion calculates the relative degrees of freedom m for n bodies connected by g joints, each with f_i degrees of freedom. This is done by summing the total degrees of freedom of the moving bodies relative to one body designated as ground, and subtracting the total degrees of freedom eliminated by the constraint of the joints,

$$m = d(n-1) - \sum_{i=1}^{g} (d - f_i) = d(n - g - 1) + \sum_{i=1}^{g} f_i \quad (46)$$

A common alternative form uses the number of independent loops l in the mechanism,

$$m = \sum_{i=1}^{g} f_i - dl \quad (47)$$

by introducing the identity

$$l = g - n + 1 \quad (48)$$

which is also one fewer than the number of branches,

$$l = p - 1 \quad (49)$$

Here it is assumed that m is both the number of degrees of freedom for the platform relative to the base and the overall relative degrees of freedom of the mechanism. This is the case for practical platform mechanisms; otherwise, there would be degrees of freedom in the mechanism that do not affect the platform-to-base motion.

Unfortunately, the mobility criterion does not take into account special geometrical arrangements of the joints that can affect the actual platform degrees of freedom. Usually $d=6$ for general spatial mechanisms and $d=3$ for planar or spherical mechanisms. However, as suggested by Waldron [45] and Hunt [46] the formula can be extended to other values less than 6 if d is taken as $\dim(\boldsymbol{S}_m)$, which is referred to here as the mechanism motion-screw system dimension. While this improves the mobility formula, it is still necessary to account for redundant constraints due to the joint arrangements that affect the mobility calculation.

The mobility criterion calculation in (46) and (47) is affected by the virtual constraint-screws $\langle \boldsymbol{S}^r_\nu \rangle$ which are usually unwittingly taken as active constraints and thus incorrectly reduce the degrees of freedom. This is compensated for by adding back in the effect of the ν virtual constraint screws,

$$m = d(n - g - 1) + \sum_{i=1}^{g} f_i + v \tag{50}$$

$$m = \sum_{i=1}^{g} f_i - dl + v \tag{51}$$

Many equivalent variations of the modified mobility criterion can be generated by introducing substitutions from (12), (13), (44), (45), (48), and (49). Perhaps the simplest forms are,

$$m = 6(n - g - 1) + \sum_{i=1}^{g} f_i + c \tag{52}$$

$$m = \sum_{i=1}^{g} f_i - 6l + c \tag{53}$$

These are just the usual $(d=6)$ Kutzbach–Grübler criterion that are adjusted by the addition of c. Therefore, all redundant constraints are taken into account by introducing c, the total number of dependent constraint screws in the platform constraint-screw multiset $\langle \boldsymbol{S}^r \rangle$. Computationally it is very efficient to calculate c from (45). The equivalent to (53) is also found in [60] as Eq. (17).

Example 3: This continues Example 1 for the 4-*RRR* mechanism in Fig. 2 to illustrate the modified mobility criterion. The mechanism constraint-screw system (26) is decomposed using (42),

$$\begin{aligned}\langle \boldsymbol{S}^r \rangle &= \{\boldsymbol{S}^r_{b1}\} + \{\boldsymbol{S}^r_{b2}\} + \{\boldsymbol{S}^r_{b3}\} + \{\boldsymbol{S}^r_{b4}\} \\ &= \{\$^r_{11}, \$^r_{12}, \$^r_{13}\} + \{\$^r_{21}, \$^r_{22}, \$^r_{23}\} + \{\$^r_{31}, \$^r_{32}, \$^r_{33}\} + \{\$^r_{41}, \$^r_{42}, \$^r_{43}\} \\ &= \underbrace{\langle \$^r_{11}, \$^r_{21}, \$^r_{31}, \$^r_{41} \rangle}_{\langle S^c \rangle} + \underbrace{\{\$^r_{12}, \$^r_{13}, \$^r_{22}, \$^r_{23}\}}_{\{S^r_c\}} + \underbrace{\langle \$^r_{32}, \$^r_{33}, \$^r_{42}, \$^r_{43} \rangle}_{\langle S^r_v \rangle}\end{aligned} \tag{54}$$

Thus there are $\nu = \text{card}\langle \boldsymbol{S}^r_\nu \rangle = 4$ virtual constraint screws. Since there are $n=10$ links and $g=12$ revolute joints the mobility calculated from Eq. (50) is $m = 5(10-12-1)+12+4=1$.

Alternatively, the platform constraint-screw multiset has $\text{card}\langle \boldsymbol{S}^r \rangle = 12$ screws and its basis $\{\boldsymbol{S}^r\}$ contains $\text{card}\{\boldsymbol{S}^r\} = 5$ screws so $c = 12-5=7$. Thus with $l=3$ loops Eq. (53) gives $m=12 - 6(3)+7=1$, which agrees.

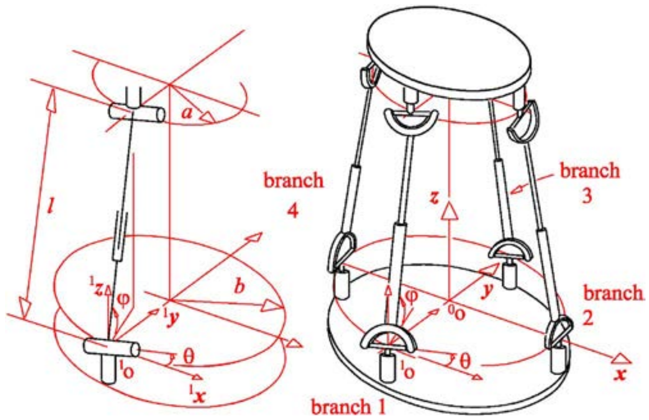

Fig. 4 4-*UPU* parallel mechanism

6 Case Studies in Mobility Analysis

In this section we provide four case studies of overconstrained mechanisms. For each case the branch motion screws are formed and then used to determine the branch constraint screws. The platform constraint-screw multiset is decomposed to determine the virtual constraint-screw multiset. Two forms of the modified mobility criterion, (50) and (53), are used to illustrate the modifications by ν and c, respectively. The platform motion screws are also determined to characterize the platform freedoms.

Example 4: In this example the mechanism constraint-screw multiset is empty. Figure 4 illustrates a four-UPU parallel mechanism.

The ends are symmetrically arranged around two reference circle of radius a at the platform and radius b at the base. Each branch consists of one prismatic joint between two universal joints. The universal joint each has a vertical and horizontal axis. During motion, the vertical axes remain vertical and the horizontal axes remain horizontal so that the platform and base are always parallel. Thus, both vertical axes have the same rotation angle and both horizontal axes have the same angular displacement.

Using a local coordinate system attached to the lower universal joint of branch 1, ${}^1O - {}^1x\,{}^1y\,{}^1z$, the five branch motion screws are

$$\{{}^1\boldsymbol{S}_{b1}\} = \left\{ \begin{array}{l} {}^1\$_{11} = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]^T, \\ {}^1\$_{12} = [\cos\theta \quad \sin\theta \quad 0 \quad 0 \quad 0 \quad 0]^T, \\ {}^1\$_{13} = [0 \quad 0 \quad 0 \quad -\cos\varphi\cos\theta \quad \cos\varphi\sin\theta \quad \sin\varphi]^T, \\ {}^1\$_{14} = [\cos\theta \quad \sin\theta \quad 0 \quad -l\sin\varphi\sin\theta \quad l\sin\varphi\cos\theta \quad -l\cos\varphi]^T, \\ {}^1\$_{15} = [0 \quad 0 \quad 1 \quad l\cos\varphi\cos\theta \quad l\cos\varphi\sin\theta \quad 0]^T \end{array} \right\} \tag{55}$$

where the first two are for the lower universal joint, the third is for the prismatic joint, and the last two are for the upper universal joint, and where the leading superscript indicates the local frame.

There is one branch constraint screw,

$$\{{}^1\boldsymbol{S}^r_{b1}\} = \{{}^1\$^r_{11} = [0 \quad 0 \quad 0 \quad \sin\theta \quad -\cos\theta \quad 0]^T\} \tag{56}$$

which is a moment normal to the axes of the universal joints.

The branch constraint screw can be transformed to the parallel global frame fixed at the center of the base by shifting along the y axis by a distance of b [69,70],

$$\$^r_{11} = ({}^0_1\mathbf{T})\,{}^1\$^r_{11} \tag{57}$$

where,

$${}^0_1\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A} & \mathbf{I} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & -b \\ 0 & 0 & 0 \\ b & 0 & 0 \end{bmatrix} \tag{58}$$

Since the four branches are arranged symmetrically in quadrants, the other three constraint screws can be produced by rotating $\$^r_{11}$ through $\alpha = \pi/2$, π, and $3\pi/2$ about the z direction,

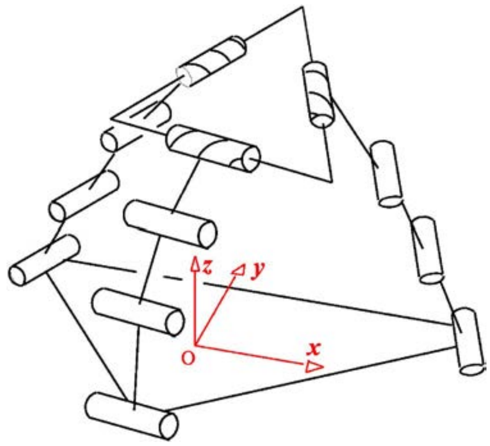

Fig. 5 3-*RRRH* parallel mechanism

$$\mathbf{T}(\alpha)\$^r_{11} = \begin{bmatrix} \mathbf{R}(\alpha) & \mathbf{0} \\ \mathbf{0} & \mathbf{R}(\alpha) \end{bmatrix} \$^r_{11}, \quad \mathbf{R}(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{59}$$

so the platform constraint-screw multiset becomes,

$$\langle \boldsymbol{S}^r \rangle = \left\langle \begin{matrix} \$^r_{11} = [0 \quad 0 \quad 0 \quad \sin\theta \quad -\cos\theta \quad 0]^T, \\ \$^r_{21} = [0 \quad 0 \quad 0 \quad -\cos\theta \quad -\sin\theta \quad 0]^T, \\ \$^r_{31} = [0 \quad 0 \quad 0 \quad -\sin\theta \quad \cos\theta \quad 0]^T, \\ \$^r_{41} = [0 \quad 0 \quad 0 \quad \cos\theta \quad \sin\theta \quad 0]^T \end{matrix} \right\rangle \tag{60}$$

A decomposition of the form (42) is,

$$\langle \boldsymbol{S}^r \rangle = \underbrace{\varnothing}_{\langle \boldsymbol{S}^c \rangle} + \underbrace{\{\$^r_{11}, \$^r_{21}\}}_{\{\boldsymbol{S}^r_c\}} + \underbrace{\langle \$^r_{31}, \$^r_{41} \rangle}_{\langle \boldsymbol{S}^r_v \rangle} \tag{61}$$

Since $\langle \boldsymbol{S}^c \rangle$ is empty then $d = \dim(\mathrm{Sm}) = 6 - \dim(\boldsymbol{S}^c) = 6$. There are $\nu = \mathrm{card}\langle \boldsymbol{S}^r_\nu \rangle = 2$ virtual constraint screws. The mechanism has $n = 10$ links and $g = 12$ joints with a total of 20 degrees of freedom so the mobility calculated from (50) is $m = 6(10 - 12 - 1) + 20 + 2 = 4$.

Alternatively, the platform constraint-screw multiset has $\mathrm{card}\langle \boldsymbol{S}^r \rangle = 4$ screws and its basis contains $\mathrm{card}\{\boldsymbol{S}^r\} = 2$ screws so $c = 4 - 2 = 2$. Thus with $l = 3$ independent loops equation (53) gives, $m = 20 - 6(3) + 2 = 4$, which agrees.

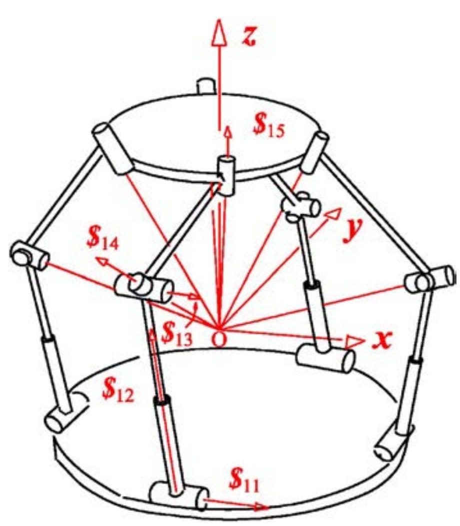

Fig. 6 4-*RPUR* parallel mechanism

The platform motion-screw system $\boldsymbol{S}_f$ is reciprocal to $\boldsymbol{S}^r$ so a basis with four screws is readily calculated as

$$\{\boldsymbol{S}_f\} = \left\{ \begin{matrix} \$_{f1} = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]^T, \\ \$_{f2} = [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0]^T, \\ \$_{f3} = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0]^T, \\ \$_{f4} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^T \end{matrix} \right\} \tag{62}$$

Hence the platform degrees of freedom are three translations and one rotation about z axis.

Example 5: This example has nonempty multisets of mechanism constraint screws and virtual constraint screws. Figure 5 illustrates a $3-RRRH$ parallel mechanism where for each branch the joints are parallel.

For branch 1 the joint axes are parallel to the x axis and the motion screws are,

$$\{\boldsymbol{S}_{b1}\} = \left\{ \begin{matrix} \$_{11} = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad r_{11}]^T, \\ \$_{12} = [1 \quad 0 \quad 0 \quad 0 \quad q_{12} \quad r_{12}]^T, \\ \$_{13} = [1 \quad 0 \quad 0 \quad 0 \quad q_{13} \quad r_{13}]^T, \\ \$_{14} = [1 \quad 0 \quad 0 \quad p_{14} \quad q_{14} \quad r_{14}]^T \end{matrix} \right\} \tag{63}$$

where p_{ij}, q_{ij}, r_{ij} are parameters determined by the positions of axes. Two linearly independent constraint screws are

$$\{\boldsymbol{S}^r_{b1}\} = \left\{ \begin{matrix} \$^r_{11} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^T, \\ \$^r_{12} = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0]^T \end{matrix} \right\} \tag{64}$$

which are two couples normal to the branch 1 axes. Similarly, the constraint screws of the remaining two branches are couples normal to their respective joints axes and can be obtained by rotations of (64) about the z direction, in a way similar to (59),

$$\{\boldsymbol{S}^r_{b2}\} = \left\{ \begin{matrix} \$^r_{21} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^T, \\ \$^r_{22} = [0 \quad 0 \quad 0 \quad c_1 \quad s_1 \quad 0]^T \end{matrix} \right\} \tag{65}$$

$$\{\boldsymbol{S}^r_{b3}\} = \left\{ \begin{matrix} \$^r_{31} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^T, \\ \$^r_{32} = [0 \quad 0 \quad 0 \quad c_2 \quad s_2 \quad 0]^T \end{matrix} \right\} \tag{66}$$

The platform constraint-screw multiset $\langle \boldsymbol{S}^r \rangle$ is formed and then decomposed using (42),

$$\begin{aligned} \langle \mathbf{S}^r \rangle &= \{\boldsymbol{S}^r_{b1}\} + \{\boldsymbol{S}^r_{b2}\} + \{\boldsymbol{S}^r_{b3}\} = \{\$^r_{11}, \$^r_{12}\} + \{\$^r_{21}, \$^r_{22}\} + \{\$^r_{31}, \$^r_{32}\} \\ &= \underbrace{\langle \$^r_{11}, \$^r_{21}, \$^r_{31} \rangle}_{\langle \boldsymbol{S}^c \rangle} + \underbrace{\{\$^r_{12}, \$^r_{22}\}}_{\{\boldsymbol{S}^r_c\}} + \underbrace{\langle \$^r_{32} \rangle}_{\langle \boldsymbol{S}^r_v \rangle} \end{aligned} \tag{67}$$

The three screws in $\langle \boldsymbol{S}^c \rangle$ are all couples in the z direction and thus two of the mechanism constraint screws are redundant. The two screws in $\{\boldsymbol{S}^r_c\}$ span all couples parallel to the x-y-plane. Finally, $\langle \boldsymbol{S}^r_\nu \rangle$ is a redundant couple in the x-y plane.

Since $\dim(\boldsymbol{S}^c) = 1$, then $d = 6 - 1 = 5$. There is $\nu = \mathrm{card}\langle \boldsymbol{S}^r_\nu \rangle = 1$ virtual constraint screws. The mechanism has $n = 11$ links and $g = 12$ single degree-of-freedom joints so the mobility calculated from (50) is $m = 5(11 - 12 - 1) + 12 + 1 = 3$.

Alternatively, the platform constraint-screw multiset has $\mathrm{card}\langle \boldsymbol{S}^r \rangle = 6$ screws and its basis contains $\mathrm{card}\{\boldsymbol{S}^r\} = 3$ screws, so $c = 6 - 3 = 3$. Thus with $l = 2$ loops, Eq. (53) gives, $m = 12 - 6(2) + 3 = 3$, which agrees.

The platform motion screws are reciprocal to $\boldsymbol{S}^r$ and a basis is given by

$$\{\boldsymbol{S}_f\} = \left\{ \begin{matrix} \$_{f1} = [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0]^T, \\ \$_{f2} = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0]^T, \\ \$_{f3} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^T \end{matrix} \right\} \tag{68}$$

so the platform has three translational degrees of freedom.

Example 6: In this example the mechanism constraint-screw multiset is empty. Figure 6 illustrates a $4-RPUR$ parallel mechanism [71].

The four branches have similar geometries. For the first branch, counting from the base: the axis of the first revolute joint $\$_{11}$ is parallel to the base; the prismatic pair $\$_{12}$ is perpendicular to two adjacent revolute axes; the horizontal axis, $\$_{13}$, of the universal joint is also parallel to the base; and the second axis, $\$_{14}$, of the universal pair and the axis of the last revolute joint $\$_{15}$ form a $2R$ spherical subchain. The two top joints of all branches together form a spherical sublinkage with all eight of the joint axes intersecting at a common point that moves with the platform motion.

Setting the origin of the reference frame at the common point, with the x axis parallel to $\$_{11}$, and the z axis normal to the base, the branch 1 motion screws are

$$\{\boldsymbol{S}_{b1}\} = \left\{\begin{array}{l} \$_{11} = [1 \quad 0 \quad 0 \quad 0 \quad q_{11} \quad r_{11}]^T, \\ \$_{12} = [0 \quad 0 \quad 0 \quad 0 \quad q_{12} \quad r_{12} \quad]^T, \\ \$_{13} = [1 \quad 0 \quad 0 \quad 0 \quad q_{13} \quad r_{13}]^T, \\ \$_{14} = [0 \quad m_{14} \quad n_{14} \quad 0 \; 0 \quad 0]^T, \\ \$_{15} = [l_{15} \quad m_{15} \quad n_{15} \quad 0 \quad 0 \quad 0]^T \end{array}\right\} \tag{69}$$

and the branch 1 constraint screw is

$$\{\boldsymbol{S}^r_{b1}\} = \{\$^r_{11} = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad]^T\} \tag{70}$$

The constraint force $\$^r_{11}$ passes through the common point and is parallel to the first revolute axis. The remaining branches are similar, so by symmetry,

$$\langle \boldsymbol{S}^r \rangle = \left\{\begin{array}{l} \$^r_{11} = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T, \\ \$^r_{21} = [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]^T, \\ \$^r_{31} = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T, \\ \$^r_{41} = [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]^T \end{array}\right\} \tag{71}$$

which is decomposed by (42) as

$$\langle \boldsymbol{S}^r \rangle = \underbrace{\varnothing}_{\langle S^c \rangle} + \underbrace{\{\$^r_{11}, \$^r_{21}\}}_{\{S^r_c\}} + \underbrace{\langle \$^r_{31}, \$^r_{41} \rangle}_{\langle S^r_\nu \rangle} \tag{72}$$

Since $\langle \boldsymbol{S}^c \rangle$ is empty then $d=6-\dim(\boldsymbol{S}^c)=6$. There are $\nu=\mathrm{card}\langle \boldsymbol{S}^r_\nu \rangle=2$ virtual constraint screws. The mechanism has $n=14$ links and $g=16$ joints with a total of 20 degrees of freedom so the mobility calculated from (50) is $m=6(14-16-1)+20+2=4$.

Alternatively, the platform constraint-screw multiset has $\mathrm{card}\langle \boldsymbol{S}^r \rangle=4$ screws and its basis contains $\mathrm{card}\{\boldsymbol{S}^r\}=2$ screws so $c=4-2=2$. Thus with $l=3$ independent loops Eq. (53) gives, $m=20-6(3)+2=4$, which agrees.

The platform motion screws are reciprocal to the $\boldsymbol{S}^r$ and a basis is given by

$$\{\boldsymbol{S}_f\} = \left\{\begin{array}{l} \$_{f1} = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T, \\ \$_{f2} = [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]^T, \\ \$_{f3} = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]^T, \\ \$_{f4} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^T \end{array}\right\} \tag{73}$$

so the platform can rotate about all axes through the common point and can translate in the z direction.

Example 7: In this example there are no mechanism constraint screws or virtual constraint screws so the modified mobility criterion reduces to the usual Kutzbach–Grübler criterion in (46) and (47) with $d=6$.

Figure 7 illustrates a symmetrical $3-RPS$ parallel mechanism with a coordinate frame attached to the platform.

Each branch has five degrees of freedom and therefore one branch constraint screw which is a force parallel to the revolute joint and intersecting the spherical joint. The platform constraint-screw multiset is,

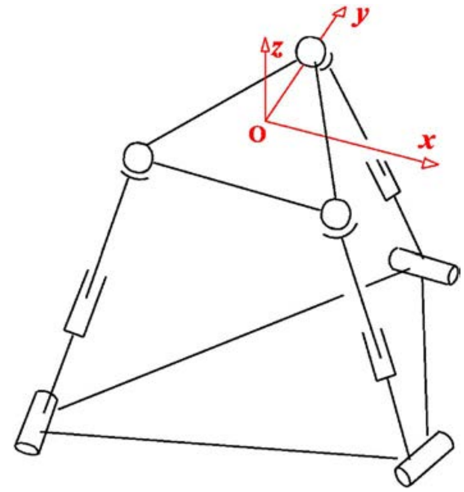

Fig. 7 3-*RPS* parallel mechanism

$$\langle \boldsymbol{S}^r \rangle = \left\{\begin{array}{l} \$^r_{11} = [l^r_{11} \quad m^r_{11} \quad 0 \quad 0 \quad 0 \quad r^r_{11}]^T, \\ \$^r_{21} = [l^r_{21} \quad m^r_{21} \quad 0 \quad 0 \quad 0 \quad r^r_{21}]^T, \\ \$^r_{31} = [l^r_{31} \quad m^r_{31} \quad 0 \quad 0 \quad 0 \quad r^r_{31}]^T \end{array}\right\}$$

which is decomposed by (42) as

$$\langle \boldsymbol{S}^r \rangle = \underbrace{\varnothing}_{\langle S^c \rangle} + \underbrace{\{\$^r_{11}, \$^r_{21}, \$^r_{31}\}}_{\{S^r_c\}} + \underbrace{\varnothing}_{\langle S^r_\nu \rangle} \tag{74}$$

Since $\langle \boldsymbol{S}^c \rangle$ is empty then $d=6-\dim(\boldsymbol{S}^c)=6$. There is $\nu=\mathrm{card}\langle \boldsymbol{S}^r_\nu \rangle=0$ virtual constraint screw. The mechanism has $n=8$ links and $g=9$ joints with a total of 15 degrees of freedom so the mobility calculated from (50) is $m=6(8-9-1)+15+0=3$.

Alternatively, the platform constraint-screw multiset has $\mathrm{card}\langle \boldsymbol{S}^r \rangle=3$ screws and its basis contains $\mathrm{cardcard}\{\boldsymbol{S}^r\}=3$ screws so $c=3-3=0$. Thus with $l=2$ independent loops Eq. (53) gives, $m=15-6(2)+0=3$, which agrees.

The platform motion-screws are reciprocal to $\boldsymbol{S}^r$ and a basis is given by

$$\{\boldsymbol{S}_f\} = \left\{\begin{array}{l} \$_{f1} = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T, \\ \$_{f2} = [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]^T, \\ \$_{f3} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^T \end{array}\right\} \tag{75}$$

so the platform can rotate about two axes through the origin that lie on the platform plane, and can translate in the z direction.

7 Conclusions

In the paper we present an original theoretical framework upon which to build a procedure to correctly evaluate the degrees of freedom for overconstrained parallel mechanisms. Screw theory and screw system analysis are used to identify and eliminate redundant constraints that do not affect platform motion. In Eq. (44) the c redundant constraints are divided into two types: (*i*) there are $(p-1)\lambda$ mechanism constraints reciprocal to all of the joint screws belonging to the same d system of screws $\boldsymbol{S}_m$ without accounting for their particular geometrical arrangement; and (*ii*) there are υ redundant constraints attributable to the particular geometrical arrangement of the joints. The latter are the virtual constraint-screws $\langle \boldsymbol{S}^r_\upsilon \rangle$ and are determined from the decomposition (42). New versions of the Kutzbach–Grübler mobility criterion are developed in which the redundant constraints are accounted for by either introducing υ in (50) and (51) or c in (52) and (53), the latter requiring fewer computations.

Extensive use is made of the dual properties of motion screws and constraint screws as exemplified by DeMorgan's laws for reciprocal systems. Further, a clear distinction is made between screw basis sets, vector spaces, and multisets to mathematically characterize various collections of screws. Detailed examples of a number of overconstrained parallel mechanisms illustrate the theory and the modified mobility formulations. The work provides a new way of evaluating both motion and constraint of overconstrained parallel mechanisms that casts new light on parallel mechanism synthesis.

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