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Architecture Singularities of Platform Manipulators

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Abstract

Based on their nature, singularities of parallel manipulators are classified into three categories, namely, architecture, configuration, and formulation singularities. The focus of this paper is on the first category, as pertaining to a class of six-dof parallel manipulators. Unlike other categories, an architecture singularity usually spans over the whole workspace or a significant part thereof, which makes very difficult to implement control or singularity-avoidance strategies. Several groups of singular architectures are reported here, with comprehensive theoretical proofs.

1 Introduction

Platform manipulators being paradigms of robotic mechanisms with parallel architecture, they have become the subject of current research recently. It turns out that most of the work related to this type of manipulators reported in the literature is on kinematics and dynamics analyses as well as control strategies—see, for example, [2],[7],[8]. However, only a few researchers have paid attention to design-related problems. Hunt studied kinematic structures of several parallel manipulators[4]; Yang and Lee investigated the kinematic feasibility of platform manipulators[10]; Fichter discussed some design problems based on his practical design work[2]. However, design-related problems are far from being totally solved. For example, existing flight simulators usually have architectures like the one shown in Fig. 1(a), namely, the joints on each of the moving and the base plates are arranged on a hexagon, its three alternating sides being equal, while the other three alternating sides being equal and much shorter than the former. A question that arises naturally is: Why such an architecture is more widespread and acceptable than others, such as the one shown in Fig. 1(b), both of whose moving and base plates are regular hexagons? Intuitively, one may think that the architecture shown

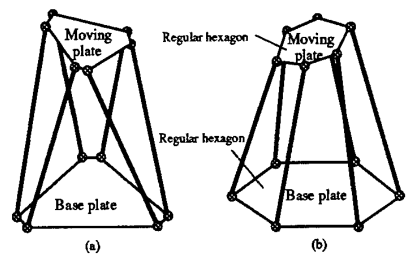

Figure 1: Platform-type parallel manipulators

in Fig. 1(a) is structurally more stable than the others because of its quasi-triangular shape. However, to the authors' knowledge, the theoretical explanation about this question is not yet available in the literature, which thus motivated the research work presented in this paper.

It is well known that there always exist singularity configurations in the workspace of a manipulator. In fact, there also exist singularities in the architecture of a parallel manipulator. In other words, some design architectures can be unstable with regard to supporting and driving the end-effector or external load. Because of its architecture-dependent nature, such a type of singularity is termed *architecture singularity* and the associated architecture is, thus, called *singular architecture*. As we will show in Section 4, for example, the architecture appearing in Fig. 1(b) is a singular architecture. In this paper, we first classify singularities of parallel manipulators into three categories such that the aforementioned architecture singularities are distinguished from the others. Then, we focus on architecture singularities as pertaining to a class of parallel manipulators by investigating the conditioning of their Jacobian matrices. It will be shown that numerical stability of the said Jacobian matrix is much dependent on the design architecture of a manipulator. With singular architectures, the Jacobian matrix becomes singular throughout the whole workspace or, at least, in a few subregions thereof. Obviously, if a design architecture falls into such a singularity, the

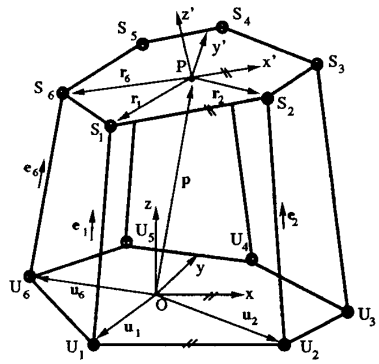

Figure 2: Illustration of position vectors

manipulator will not be able to work because it fails to balance the load on its moving plate. In other words, it fails to transfer a full wrench from the actuated joints to the end-effector. However, even if an architecture does not fall into a singularity, it could still be ill-conditioned, which leads to a very poor motion and force-transmission performance. Therefore, a comprehensive study of architecture singularity is warranted, especially from the design point of view.

2 Jacobian Matrix

The type of parallel manipulators under study consists of two plates and six extendible legs. One of the plates, called the *moving plate*, is considered the end-effector, which undergoes a six-dof motion. The other plate is fixed on the ground and hence, is called the *base plate*. With no loss of generality, each of the two plates is assumed to have six joints distributed at the vertices of an *arbitrary* hexagon. The six joints on the moving plate are usually spherical and indicated by S_1 through S_6, while the joints on the base plate are universal and indicated by U_1 through U_6, as shown in Fig. 2. The leg connecting joints U_i and S_i is referred to as the ith leg. For the description of the relative configuration of the two plates, a base coordinate frame $\mathcal{B}(O, x, y, z)$ and a moving frame $\mathcal{M}(P, x', y', z')$ are defined on the two plates, respectively, as shown in Fig. 2. The origins of the two frames are located at the centers of the corresponding plates. Moreover, it is assumed that all vectors and matrices appearing in this paper are represented in the base frame $\mathcal{B}$ unless otherwise indicated.

With the above definitions, the configuration of the moving plate with respect to the base plate can be described by a vector $\mathbf{p}$ directed to point P from O, and a rotation matrix $\mathbf{Q}$ representing the orientation of the moving frame $\mathcal{M}$ with respect to the base frame $\mathcal{B}$. The geometries of the two plates can be described by vectors $\mathbf{u}_i$ and $\mathbf{r}_i$ here $i = 1, 2, \cdots, 6$, as indicated in Fig. 2. Clearly, each of the two vector sets $\{\mathbf{u}_i\}_1^6$ and $\{\mathbf{r}_i\}_1^6$ is coplanar. Moreover, vector $\mathbf{u}_i$ has constant components in frame $\mathcal{B}$, while the components of vector $\mathbf{r}_i$ are orientation-dependent in $\mathcal{B}$, i.e.,

$$\mathbf{r}_i = \mathbf{Q}\mathbf{r}'_i \tag{1}$$

where $\mathbf{r}'_i$ is the representation of vector $\mathbf{r}_i$ in the moving frame $\mathcal{M}$. Apparently, $\mathbf{r}'_i$ has constant components since $\mathcal{M}$ moves with the moving plate. With the above definitions, it is a simple matter to derive basic kinematic constraints of the manipulator, namely,

$$(\mathbf{p} + \mathbf{r}_i - \mathbf{u}_i)^T(\mathbf{p} + \mathbf{r}_i - \mathbf{u}_i) = q_i^2, \tag{2}$$

for $i = 1, 2, \cdots, 6$. Here q_i denotes the joint coordinate representing the length of the ith leg. Besides, let $\dot{\mathbf{p}}$ denote the 3-D velocity vector of reference point P, $\boldsymbol{\omega}$ denote the 3-D angular velocity vector of the moving plate, and $\dot{q}_i$ denote joint velocity associated with the ith leg. Furthermore, to treat the orientation and translation in the same frame, a 6-D twist vector of the end-effector is defined as

$$\mathbf{t} = \begin{bmatrix} \boldsymbol{\omega} \\ \dot{\mathbf{p}} \end{bmatrix}. \tag{3}$$

Upon differentiating the constraint equations (2) with respect to time, one can derive the desired relation between the end-effector twist and the joint velocities, that takes on the form

$$\mathbf{A}\mathbf{t} = \mathbf{B}\dot{\mathbf{q}} \tag{4}$$

where

$$\dot{\mathbf{q}} = [\,\dot{q}_1 \quad \dot{q}_2 \quad \cdots \quad \dot{q}_6\,]^T, \tag{5a}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{r}_1 \times (\mathbf{p} - \mathbf{u}_1) & \cdots & \mathbf{r}_6 \times (\mathbf{p} - \mathbf{u}_6) \\ \mathbf{r}_1 + \mathbf{p} - \mathbf{u}_1 & \cdots & \mathbf{r}_6 + \mathbf{p} - \mathbf{u}_6 \end{bmatrix}^T, \tag{5b}$$

$$\mathbf{B} = \mathrm{diag}(q_1, q_2, \cdots, q_6). \tag{5c}$$

Based on eq. (4), the solution of the *inverse* velocity problem can be expressed as $\dot{\mathbf{q}} = \mathbf{B}^{-1}\mathbf{A}\mathbf{t}$, while the solution of the *direct* problem is defined as $\mathbf{t} = \mathbf{A}^{-1}\mathbf{B}\dot{\mathbf{q}}$. Obviously, the inversion of the diagonal matrix $\mathbf{B}$ is very simple and always possible, unless one of the leg lengths vanishes, which is impossible in practice. In other words, the inverse kinematics for this type of manipulators is trivial. However, the inversion of matrix $\mathbf{A}$ is not straightforward and hence, it deserves further study. It is pointed out that the Jacobian matrix of a serial manipulator is defined based on the linear transformation from the joint velocity vector to the end-effector twist vector. If the same definition is adopted here, the resulting Jacobian matrix will be $\mathbf{A}^{-1}\mathbf{B}$, which is inconvenient for analysis because of the presence of matrix inversion in the definition itself. Apparently, with such a definition, the Jacobian matrix is undefined when matrix $\mathbf{A}$ is singular. Instead, we define the Jacobian matrix of the manipulators under study as the mapping from the end-effector twist vector to the joint velocity vector, namely,

$$\mathbf{J}\mathbf{t} = \dot{\mathbf{q}}, \tag{6}$$

where $\mathbf{J}$ denotes the aforementioned 6×6 Jacobian matrix. Note that $\mathbf{J}$ exists, even in the presence of singularities of $\mathbf{A}$. From eq. (4), the Jacobian matrix can be expressed as

$$\mathbf{J} = \mathbf{B}^{-1}\mathbf{A}. \tag{7}$$

Let $\mathbf{e}_i$ denote the unit vector along the joint axis of the ith leg, for $i = 1, 2, \cdots, 6$, as shown in Fig. 2. It can then be shown that the Jacobian matrix takes on a simple form [5], namely,

$$\mathbf{J} = \begin{bmatrix} \mathbf{r}_1 \times \mathbf{e}_1 & \mathbf{r}_2 \times \mathbf{e}_2 & \cdots & \mathbf{r}_6 \times \mathbf{e}_6 \\ \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_6 \end{bmatrix}^T . \tag{8}$$

From the above discussion, it is apparent that solving for the twist $\mathbf{t}$ from eq. (6) is either impossible or inaccurate if $\mathbf{J}$ is singular or ill-conditioned, respectively. Similarly, a singular $\mathbf{J}$ will cause force transmission problems [5],[6].

3 Singularity Classification

Gosselin and Angeles [3] classified singularities of manipulators into three types, based on the combination of singularities of matrices $\mathbf{A}$ and $\mathbf{B}$. That classification is comprehensive from the mathematical point of view, and hence is suitable for implementing algorithms of singularity detection in some applications. In fact, different singularities are different in nature. For instance, some singularities depend only on individual configurations. These singularities disappear if the associated configurations change. Other singularities are caused by manipulator architectures and hence, changing configuration may not help remove such a singularity. However, from a mathematical point of view, both singularities may lead to a singular $\mathbf{A}$ or $\mathbf{B}$ or both. It follows that a classification based only on the singularities of matrices $\mathbf{A}$ and $\mathbf{B}$ does not reflect the nature of singularities. On the other hand, it is understandable that different singularities with different nature may need different treatment in practice. For example, an architecture singularity can be eliminated only by design, while other singularities cannot. We present here a different classification of singularities of parallel manipulators. This classification consists of three categories, as described below:

a) **Architecture singularity:**
A singularity which is caused by a particular architecture of a manipulator. Such a singularity exists for all configurations inside the whole or a part of the manipulator workspace. An example of this type of singularity is shown in Fig. 1(b), as we will prove in Section 4.

b) **Configuration singularity:**
This is a singularity caused by a particular configuration of a manipulator, and hence, it depends only on one individual configuration. An example of such a singular configuration of the platform manipulator arises when the moving plate is parallel to the base plate and is oriented with respect to the latter by a rotation through an angle of $\pi/2$ about the common normal of the two plates [6].

c) **Formulation singularity:**
A singularity which is caused by the failure of a kinematic model at a particular configuration of a manipulator. For example, if Euler 3-2-1 angles are employed to represent the orientation of the moving plate, the associated kinematic model will become singular if the second Euler angle equals $\pm\pi/2$ [9].

Among the aforementioned three types of singularities, the architecture singularity is the worst case because a manipulator falling into such a singularity fails to work in the whole or most of its workspace. Hence, such a singularity must be avoided by design. On the other hand, the formulation singularity is associated with particular formulation methods. It can be eliminated by changing the kinematic model and hence, it is always avoidable. The configuration singularity, in turn, depends only on individual configurations and hence, its avoidance is possible at the trajectory-planning stage. The three types of singularities obey certain relations, namely, the architecture singularity overrides the configuration singularity and the configuration singularity overrides the formulation singularity. In general, all three types of singularities will involve a singularity of matrix $\mathbf{A}$ or $\mathbf{B}$ or both, because both matrices may be functions not only of joint variables but also of design variables. This is why the aforementioned different singularities cannot be distinguished from each other by just testing each of these matrices for singularity. However, for the manipulators under study, only a configuration singularity may yield a singular $\mathbf{B}$ because this matrix involves only joint variables.

In the literature, most of the research work on singularity issues are related to configuration singularities. Intensive studies on this type of singularity have been presented in [3],[6]. Formulation singularities have also been reported in [1],[9]. Architecture singularities, on the contrary, have received very little attention in the literature. Nevertheless, the third type of singularity in Gosselin and Angeles' classification [3] can be interpreted as a case of the architecture singularity because of its design-dependence feature. However, since that type of singularity is limited to the case in which matrices $\mathbf{A}$ and $\mathbf{B}$ are simultaneously singular, it is only a subgroup of the architecture singularity. In practice, architecture singularities are avoided by design, either consciously or unconsciously. However, it should be pointed out that even if a manipulator architecture does not fall into this category, it may still be ill-conditioned if this architecture is close to a singular one. As a consequence, the performance of such a manipulator is very poor. Detailed work on the issue of ill-conditioning and the corresponding design considerations will be reported in an upcoming paper. It is apparent that a thorough understanding of architecture singularities is a basis for avoiding these singularities and corresponding ill-conditioning in manipulator design.

4 Architecture Singularities

By close inspection of various numerical results of the direct kinematics from computer simulations with different architectures of platform manipulators, we obtained the following interesting observations: 1) The Jacobian matrix $\mathbf{J}$ is singular throughout the whole workspace if both the moving and the base plates are similar and regular polygons; 2) The Jacobian matrix $\mathbf{J}$ is singular in most subregions of the workspace if both the moving and the base plates are similar irregular polygons and each pair of their corresponding vertices is connected by a leg.

A few singular architectures are shown in Fig. 3, where numbers at the vertices of each pair of plates indicate the leg connections. Among the three architectures shown in that figure, the first one is an example of observation 1) and the other two are the cases of observation 2).

We show below that the above two observations can be stated as theorems. Before proving these theorems, we emphasize here that

$$\text{rank}(\mathbf{J}) = \text{rank}(\mathbf{B}^{-1}\mathbf{A}) = \text{rank}(\mathbf{A}), \qquad (9)$$

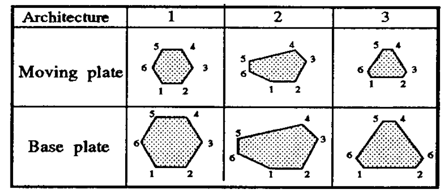

Figure 3: Examples of singular architectures

since matrix $\mathbf{B}$, as expressed in eq. (5c), is always of full rank unless one of the legs has zero length, which is not the case in practice. Moreover, we define

$$\mathbf{A} \equiv \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}^T \qquad (10)$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are 3×6 submatrices derived from eq. (5b), namely,

$$\mathbf{A}_1 = [\mathbf{r}_1 \times (\mathbf{p} - \mathbf{u}_1) \quad \cdots \quad \mathbf{r}_6 \times (\mathbf{p} - \mathbf{u}_6)], \qquad (11)$$

$$\mathbf{A}_2 = [\mathbf{r}_1 + \mathbf{p} - \mathbf{u}_1 \quad \cdots \quad \mathbf{r}_6 + \mathbf{p} - \mathbf{u}_6]. \qquad (12)$$

Theorem 1

The Jacobian matrix $\mathbf{J}$ is singular throughout the whole workspace if both the moving and the base plates are similar and regular polygons.

This theorem is proven by applying elementary column operations to matrix $\mathbf{A}$. The architecture condition involved here leads to a set of symmetry relations, i.e.,

$$\mathbf{u}_i = -\mathbf{u}_{i+3}, \qquad \mathbf{r}_i = -\mathbf{r}_{i+3}, \qquad \text{for } i = 1,\, 2,\, 3. \quad (13)$$

From the above relations, we obtain

$$\mathbf{r}_i \times (\mathbf{p} - \mathbf{u}_i) + \mathbf{r}_{i+3} \times (\mathbf{p} - \mathbf{u}_{i+3}) = -2\mathbf{r}_i \times \mathbf{u}_i \quad (14a)$$

$$(\mathbf{r}_i + \mathbf{p} - \mathbf{u}_i) + (\mathbf{r}_{i+3} + \mathbf{p} - \mathbf{u}_{i+3}) = 2\mathbf{p} \qquad (14b)$$

Since the left-hand side of eqs. (14) is the summation of the ith and $(i+3)$th columns of matrix $\mathbf{A}$, we can infer that $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}')$ where

$$\mathbf{A}' = \begin{bmatrix} \mathbf{r}_1 \times (\mathbf{p} - \mathbf{u}_1) & \mathbf{r}_2 \times (\mathbf{p} - \mathbf{u}_2) & \mathbf{r}_3 \times (\mathbf{p} - \mathbf{u}_3) & \mathbf{r}_1 \times \mathbf{u}_1 & \mathbf{r}_2 \times \mathbf{u}_2 & \mathbf{r}_3 \times \mathbf{u}_3 \\ \mathbf{r}_1 + \mathbf{p} - \mathbf{u}_1 & \mathbf{r}_2 + \mathbf{p} - \mathbf{u}_2 & \mathbf{r}_3 + \mathbf{p} - \mathbf{u}_3 & -\mathbf{p} & -\mathbf{p} & -\mathbf{p} \end{bmatrix}. \quad (15)$$

Next, adding the last three columns to the corresponding first three columns of matrix $\mathbf{A}'$, we derive that $\text{rank}(\mathbf{A}') = \text{rank}(\mathbf{A}'')$ where

$$\mathbf{A}'' = \begin{bmatrix} \mathbf{r}_1 \times \mathbf{p} & \cdots & \mathbf{r}_3 \times \mathbf{p} & \mathbf{r}_1 \times \mathbf{u}_1 & \cdots & \mathbf{r}_3 \times \mathbf{u}_3 \\ \mathbf{r}_1 - \mathbf{u}_1 & \cdots & \mathbf{r}_3 - \mathbf{u}_3 & -\mathbf{p} & \cdots & -\mathbf{p} \end{bmatrix}.$$

Moreover, from the geometry of a regular hexagon, we have additional relations, namely,

$$\mathbf{u}_3 = -\mathbf{u}_1 + \mathbf{u}_2, \qquad \mathbf{r}_3 = -\mathbf{r}_1 + \mathbf{r}_2. \tag{16}$$

Now, adding the first two columns to the third column of matrix $\mathbf{A}''$ and applying the relations given in eq. (16), we infer that

$$\begin{aligned}\text{rank}(\mathbf{A}'') &= \text{rank}\begin{bmatrix} \mathbf{r}_1 \times \mathbf{p} & \mathbf{r}_2 \times \mathbf{p} & \mathbf{0} \\ \mathbf{r}_1 - \mathbf{u}_1 & \mathbf{r}_2 - \mathbf{u}_2 & \mathbf{0} \end{bmatrix} \\ &\qquad \begin{bmatrix} \mathbf{r}_1 \times \mathbf{u}_1 & \mathbf{r}_2 \times \mathbf{u}_2 & \mathbf{r}_3 \times \mathbf{u}_3 \\ -\mathbf{p} & -\mathbf{p} & -\mathbf{p} \end{bmatrix}\end{aligned} \tag{17}$$

which is obviously singular. This means that matrix $\mathbf{A}$ and, therefore, the Jacobian matrix $\mathbf{J}$ are also singular. Notice that vector $\mathbf{r}_i$, as expressed in eq. (1), is orientation-dependent. In the above proof, no specific position $\mathbf{p}$ nor specific orientation $\mathbf{Q}$ is assumed and hence, eq. (17) holds for any $\mathbf{p}$ and any $\mathbf{Q}$. In other words, the Jacobian matrix is singular at any configuration in the whole workspace.

Theorem 2

The Jacobian matrix $\mathbf{J}$ is singular in a few significant subregions of the workspace if both the moving and the base plates are similar and irregular polygons and each pair of their corresponding vertices is connected by a leg.

Since this theorem involves singularities in subregions of the workspace, the corresponding proofs will be done case by case. Each case involves one subregion.

Case 1: *The Jacobian matrix is singular in the subregion of the workspace where the centers of both plates are coincident.*

The given subregion can be expressed as $\mathbf{p} = \mathbf{0}$, which leads to

$$\mathbf{A}_2 = [\mathbf{r}_1 - \mathbf{u}_1 \quad \mathbf{r}_2 - \mathbf{u}_2 \quad \cdots \quad \mathbf{r}_6 - \mathbf{u}_6]. \tag{18}$$

Moreover, from the architecture condition, we have

$$\mathbf{r}_i = \alpha \mathbf{Q} \mathbf{u}_i, \qquad \text{for } i = 1, 2, \cdots, 6, \tag{19}$$

where α is a nonzero scalar representing the scaling factor between the two similar polygonal plates. Substituting eq. (19) into matrix eq. (18), we derive

$$\begin{aligned}\mathbf{A}_2 &= [\alpha \mathbf{Q} \mathbf{u}_1 - \mathbf{u}_1 \quad \cdots \quad \alpha \mathbf{Q} \mathbf{u}_6 - \mathbf{u}_6] \\ &= [\alpha \mathbf{Q} - \mathbf{1}][\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_6].\end{aligned} \tag{20}$$

Hence,

$$\text{rank}(\mathbf{A}_2) = \text{rank}\Big([\alpha \mathbf{Q} - \mathbf{1}][\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_6]\Big) \tag{21}$$

from which we know rank($\mathbf{A}_2$) is equal to the smaller of rank($\mathbf{Q} - \mathbf{1}$) and rank($[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_6]$). Obviously, matrix $[\alpha \mathbf{Q} - \mathbf{1}]$ is configuration dependent and hence, its rank may vary from configuration to configuration. However, since all vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_6$ are coplanar,

$$\text{rank}[\mathbf{u}_i \quad \mathbf{u}_j \quad \mathbf{u}_k] < 3, \tag{22}$$

for $i, j, k = 1, 2, \cdots, 6$. This means that

$$\text{rank}([\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_6]) < 3, \tag{23}$$

and hence,

$$\text{rank}(\mathbf{A}_2) < 3. \tag{24}$$

Thus, we can conclusion that

$$\text{rank}(\mathbf{J}) = \text{rank}(\mathbf{A}) \leq \text{rank}(\mathbf{A}_1) + \text{rank}(\mathbf{A}_2) < 6. \tag{25}$$

In the above proof, no specific orientation $\mathbf{Q}$ is assumed and therefore, the result is valid for any orientation of the moving plate.

Case 2: *The Jacobian matrix is singular in the subregion of the workspace where both plates have the same orientation.*

Because of no relative orientation between two plates, $\mathbf{Q}$ is the identity matrix, which leads to

$$\mathbf{r}_i = \alpha \mathbf{u}_i, \qquad \text{for } i = 1, 2, \cdots, 6. \tag{26}$$

Substituting relations (26) into the expression of matrix $\mathbf{A}_1$, as given in eq. (11), we obtain

$$\begin{aligned}\mathbf{A}_1 &= [\alpha \mathbf{u}_1 \times \mathbf{p} \quad \alpha \mathbf{u}_2 \times \mathbf{p} \quad \cdots \quad \alpha \mathbf{u}_6 \times \mathbf{p}] \\ &= -\alpha [\mathbf{P} \mathbf{u}_1 \quad \mathbf{P} \mathbf{u}_2 \quad \cdots \quad \mathbf{P} \mathbf{u}_6] \\ &= -\alpha \mathbf{P} [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_6]\end{aligned} \tag{27}$$

where $\mathbf{P}$ is the 3×3 matrix defined such that

$$\mathbf{P} \mathbf{u}_i = \mathbf{p} \times \mathbf{u}_i. \tag{28}$$

It can be readily shown that both matrices $\mathbf{P}$ and $[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_6]$ are rank-deficient because $\mathbf{P}$ is skew-symmetric and all vectors $\mathbf{u}_i$ are coplanar. Therefore, $\mathbf{A}_1$ is also rank-deficient, which leads to

$$\text{rank}(\mathbf{J}) = \text{rank}(\mathbf{A}) \leq \text{rank}(\mathbf{A}_1) + \text{rank}(\mathbf{A}_2) < 6. \tag{29}$$

Because no particular $\mathbf{p}$ is assumed in the above proof, the conclusion is valid for any position of the moving plate.

Case 3: *The Jacobian matrix is singular in the subregion of the workspace where the moving and the base plates are parallel.*

This case is proven by applying basic row operations to matrix $\mathbf{A}$. For simplicity, we ignore the third row of matrix $\mathbf{A}$ and assume that it always has full rank. This assumption will not affect the final result because none of the row operations involved in the proof procedure will use the third row. Based on the above assumption and the fact that the two plates are parallel, the rank of $\mathbf{A}$ can be expressed as

$$\text{rank}(\mathbf{A}) = \text{rank}\begin{bmatrix} \mathbf{r}_1 \times \mathbf{p} & \cdots & \mathbf{r}_6 \times \mathbf{p} \\ \mathbf{p} + \mathbf{r}_1 - \mathbf{u}_1 & \cdots & \mathbf{p} + \mathbf{r}_6 - \mathbf{u}_6 \end{bmatrix}$$

where product $\mathbf{r}_i \times \mathbf{u}_i$ has been deleted because it contributes to the third row only. Let p_z denote the third component of vector $\mathbf{p}$. Then, the rank of $\mathbf{A}$ can be expressed as

$$\text{rank}(\mathbf{A}) = \text{rank}\begin{bmatrix} p_z\mathbf{r}_1 & \cdots & p_z\mathbf{r}_6 \\ \mathbf{p} + \mathbf{r}_1 - \mathbf{u}_1 & \cdots & \mathbf{p} + \mathbf{r}_6 - \mathbf{u}_6 \end{bmatrix}$$

which can be further expressed as:

$$\text{rank}(\mathbf{A}) = \text{rank}\begin{bmatrix} p_z\mathbf{r}_1 & p_z\mathbf{r}_2 & \cdots & p_z\mathbf{r}_6 \\ \mathbf{p} - \mathbf{u}_1 & \mathbf{p} - \mathbf{u}_2 & \cdots & \mathbf{p} - \mathbf{u}_6 \end{bmatrix}.$$

Moreover, since the two plates are similar, vector $\mathbf{u}_i$ can always be expressed as a linear combination of the components of vector $\mathbf{r}_i$, as expressed in eq. (19). Therefore, we finally have

$$\text{rank}(\mathbf{A}) = \text{rank}\begin{bmatrix} p_z\mathbf{r}_1 & p_z\mathbf{r}_2 & \cdots & p_z\mathbf{r}_6 \\ \mathbf{p} & \mathbf{p} & \cdots & \mathbf{p} \end{bmatrix} \quad (30)$$

which clearly shows that the rank of the last three rows is equal to unity, because these rows are proportional to each other. As a result, the rank of the original matrix $\mathbf{A}$ is, thus, not greater than four. It follows that, based on eq. (9), the Jacobian matrix is singular. In the above proof, the position vector $\mathbf{p}$ remains arbitrary, but the rotation $\mathbf{Q}$ is limited to the case in which the two plates are parallel.

Finally, we point out that the cases studied above have already shown that the subregions of the workspace where the Jacobian matrix is singular are significant if the architecture of a manipulator is singular. In other words, these subregions are not only very large, but also most commonly used for a manipulator in practice. For example, the home configuration is included in Case 2 or Case 3.

5 Conclusions

Based on their nature, the singularities of parallel manipulators are classified into three categories, namely, the architecture singularity, the configuration singularity and the formulation singularity. The focus has been on the architecture singularity of platform manipulators. It was shown that an architecture singularity usually spans the whole workspace or several significant subregions of it, rather than appearing at isolated configurations. Several cases of singular architectures have been studied. Comprehensive theoretical proofs were presented to support the study of the aforementioned singular architectures. This research work can serve as a guide for the optimum design of parallel manipulators in the sense of achieving optimum kinematic and force-transmission performances.

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