Conditions for Line-Based Singularities in Spatial Platform Manipulators

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Received January 12, 1997; accepted September 5, 1997

In this paper, we consider singular configurations of a robotic system that can be modeled as a platform supported by serial robotic chains. The goal is to determine the conditions under which these singular configurations depend purely on the geometry of lines associated with each supporting chain. These conditions are satisfied by many manipulator systems, particularly those based on the Stewart platform. A geometric classification of these ''line-based'' singularities is presented. *© 1998 John Wiley & Sons, Inc.*

この発表では、直列のロボット・チェインで支えられているプラットフォームとしてモデル化できるロボット・システムについて、その特異配置を考察する。今回の研究の目的は、特異配置が、プラットフォームを支えている各チエインに関する列の幾何学的配置だけに依存するようになる条件を特定することである。これらの条件は、多くのマニピュレータ・システムにおいて満たすことが可能であり、特にスチュワート・プラットフォームを基にしている場合がそうである。そして、このような列に基づいた特異性の幾何学的分類について説明する。

Journal of Robotic Systems 15(1), 43–55 (1998)

CCC 0741-2223/98/010043-13

1. INTRODUCTION

This paper studies the Jacobian of robotic systems that consist of a platform supported by serial chain manipulators. The primary focus is on those designs that have Jacobians consisting purely of lines and the conditions under which these lines become linearly dependent. Platforms supported by serial chains are known as *parallel* manipulators and appear as walking machines,[1] mechanical hands manipulating an object,[2] and special systems such as vehicle simulator platforms.[3,4] See Figure 1. The Jacobian of these systems defines the contribution of each actuator to the resultant force and torque supporting the platform.[5] If this contribution is always a pure force for every actuator, then the columns of the Jacobian are the Plücker coordinates of lines in space; and singular configurations of the platform manipulator are associated with linearly dependent sets of lines, which we term *line-based singularities*.

Merlet[6] introduced a classification of line-based singularities drawn from the work Dandurand[7] and others, in order to study the singularities of a "triangular simplified symmetric manipulator." He was able to identify these singular configurations by inspection of the various ways the axes of the linear actuators could form linearly dependent sets of lines. A similar analysis was used Notash and Podhorodeski[8] to study three branch parallel manipulators, and by Collins and Long[9] to analyze a pantagraph-based hand-controller. This geometric method of analysis by-passes the computation of the determinant of the Jacobian matrix, which is used to determine singular configurations in serial chain manipulators.[11,12]

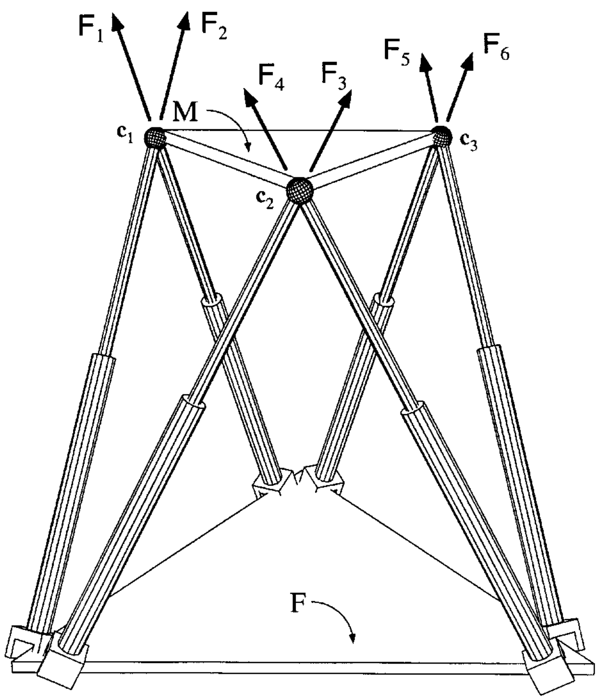

Figure 1. The Stewart platform is an example of a platform manipulator. Singular configurations occur when the lines $\mathbf{F}_i$ become linearly dependent.

Our goal in this paper is to determine the design features of a platform manipulator that ensure that only line-based singularities exist. This is equivalent to the characteristic that each supporting chain of the system can apply only a pure force to the platform. We then consider in detail each of the linearly dependent sets of lines that can arise. This study draws on classical results of line geometry found in Jessop[13], Salmon,[14] and Woods,[15] but tailored to the characteristics of these physical systems.

These results are a special case of a broader study of six-vectors known as *screws* introduced by Ball.[16] Hunt[17] provides a survey of linearly dependent sets of screws important to mechanism theory. Gibson and Hunt[18,19] and Martinez and Duffy[20,21] provide a more detailed look at these "screw systems." Philips[22,23] provides a machine based perspective of screw theory.

Merlet's geometric approach to the singularity analysis of platform manipulator is a powerful and elegant tool. While the general cases of linearly dependent sets of lines are challenging to visualize, the special cases can be evaluated by inspection. A goal of this article is to make this technique more widely known.

2. SCREW THEORY

The angular velocity **w** and linear velocity **v** of a moving body are three-dimensional vectors that can be assembled into a six vector called a *twist*. Similarly, the resultant force **f** and torque **t** acting at a point on the body can be assembled into another six vector called a *wrench*. See Figure 2. The mathematics of these these vector pairs, also called *dual vectors*, is known as *screw theory* (see refs. 16, 17, 22, and 23). A useful mathematical formalism known as a *dual vector algebra* is described in Dimentberg,[24] Woo and Freudenstein,[25] and Bottema and Roth.[26]

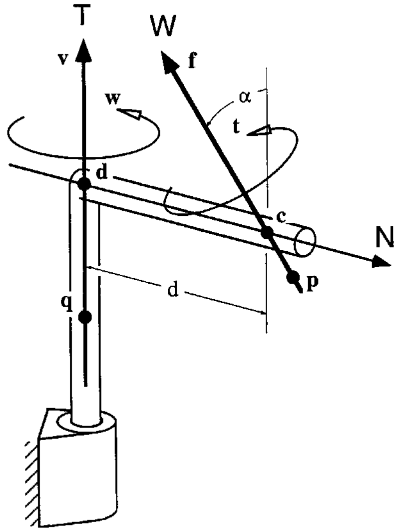

Figure 2. The wrench $\mathbf{W}=(\mathbf{f}, \mathbf{p}\times\mathbf{f}+\mathbf{t})$ acts on a rigid link which is constrained to move with twist $\mathbf{T}=(\mathbf{w}, \mathbf{q}\times\mathbf{q}+\mathbf{v})$.

Here we present a brief summary of the results needed to define the Jacobian of a parallel manipulator system.

2.1. Twists, Wrenches, and Screws

2.1.1. Twists

Consider the position and orientation of a frame M in the end-effector of a serial chain robot defined by the 4×4 homogeneous transformation, $[T(\bar{\theta})]$:

$$[T(\bar{\theta})]=\begin{bmatrix} A(\bar{\theta}) & \mathbf{d}(\bar{\theta}) \\ 0\ 0\ 0 & 1 \end{bmatrix}. \tag{1}$$

The 3×3 rotation $[A(\bar{\theta})]$ and 3×1 translation $\mathbf{d}(\bar{\theta})$ define the position of M in a fixed frame F in terms of $\bar{\theta}=(\theta_1, \theta_2, \dots, \theta_6)^T$, the vector of joint parameters that define the configuration of the chain.

The angular and linear velocity of the end-effector are computed by constructing what Murray et al.[27] call the *Lie algebra element*, and McCarthy[28] calls the *tangent operator*, $[S]$, associated with motion $[T(t)]$:

$$[S]=[\dot{T}(t)][T(t)]^{-1}=\begin{bmatrix} \dot{A}A^T & -\dot{A}A^T\mathbf{d}+\dot{\mathbf{d}} \\ 0\ 0\ 0 & 0 \end{bmatrix}. \tag{2}$$

The components of the 3×3 skew-symmetric matrix $[\Omega]=[\dot{A}A^T]$ can be assembled into the angular velocity vector $\mathbf{w}$, and $\mathbf{v}=\dot{\mathbf{d}}(t)$ is the *linear velocity vector*. The six vector $\mathsf{T}=(\mathbf{w}, \mathbf{q}\times\mathbf{w}+\mathbf{v})$ is called the *twist* of the motion. Equation (2) can be expanded in terms of the partial twists S_j which are computed from the 4×4 matrices $(\partial T/\partial\theta_j)T^{-1}$ associated with each joint parameter θ_j. The twist, T, of the end-effector of the serial chain is related to the partial twists S_j by the equation:

$$\mathsf{T}=\begin{bmatrix} \mathbf{w} \\ \mathbf{q}\times\mathbf{w}+\mathbf{v} \end{bmatrix}=[\mathsf{S}_1, \mathsf{S}_2, \dots, \mathsf{S}_6]\dot{\bar{\theta}}. \tag{3}$$

Each of the m serial chains supporting a platform manipulator has the platform as its end-effector; therefore, each chain contributes to the same twist. Let $\mathsf{S}_j^{(i)}$ be the partial screw on the ith chain. Then, we have the m equations:

$$\mathsf{T}^{(i)}=\begin{bmatrix} \mathbf{w} \\ \mathbf{q}\times\mathbf{w}+\mathbf{v} \end{bmatrix}=\left[\mathsf{S}_1^{(i)}, \mathsf{S}_2^{(i)}, \dots, \mathsf{S}_6^{(i)}\right]\dot{\bar{\theta}}^{(i)}, \quad i=1, \dots, m. \tag{4}$$

Phillips[22,23] uses the terminology *motion screw* for a twist. The term *joint screw* is also often used to describe the instantaneous movement allowed by a joint.

2.1.2. Wrenches

The resultant force and torque, $\mathbf{f}$ and $\mathbf{t}$, exerted at a point $\mathbf{p}$ on an end-effector by the actuators of a serial chain can be assembled into the screw W given by

$$\mathsf{W}=\begin{Bmatrix} \mathbf{f} \\ \mathbf{p}\times\mathbf{f}+\mathbf{t} \end{Bmatrix}, \tag{5}$$

called a *wrench*. The total wrench W applied to a platform supported by m serial chains is the sum of the individual wrenches:

$$\mathsf{W}=\sum_{i=1}^{m}\mathsf{W}^{(i)}. \tag{6}$$

We use the term *actuator screw* for wrenches that represent the force/torque contribution of an actuator. Philips[22,23] refers to a normalized version of this wrench as an *action screw*.

2.1.3. Screws

Associated with a screw $\mathsf{S}=(\mathbf{s}, \mathbf{u})$, which may be a wrench or a twist, is a line $L(t)=\mathbf{r}+t\mathbf{s}$ called the *axis* of the screw. A point $\mathbf{r}$ on this axis is obtained from the two vectors of S using the formula:

$$\mathbf{r} = \frac{\mathbf{s} \times \mathbf{u}}{\mathbf{s} \cdot \mathbf{s}}. \quad (7)$$

The Plücker coordinates of this line, $\mathsf{L} = (\mathbf{s}, \mathbf{r} \times \mathbf{s})$, form a particular type of screw that we call a *Plücker vector*. Notice that the two vectors that make up a Plücker vector are, by definition, orthogonal.

A screw S is written in terms of the Plücker coordinates of its axis, as

$$\mathsf{S} = \begin{Bmatrix} \mathbf{s} \\ \mu\mathbf{s} + \mathbf{r} \times \mathbf{s} \end{Bmatrix} \qquad \text{where } \mu = \frac{\mathbf{s} \cdot \mathbf{u}}{\mathbf{s} \cdot \mathbf{s}}. \quad (8)$$

The parameter μ, called the *pitch* of the screw, is the ratio of the magnitude of the component of $\mathbf{u}$ in the direction $\mathbf{s}$ to the magnitude of $\mathbf{s}$.

2.2. Virtual Work

The work done by a wrench $\mathsf{W} = (\mathbf{f}, \mathbf{p} \times \mathbf{f} + \mathbf{t})$ as it moves through a twist $\mathsf{T} = (\mathbf{w}, \mathbf{q} \times \mathbf{w} + \mathbf{v})$ over a virtual time period δt is given by

$$\begin{aligned} \delta\mathscr{W} &= (\mathbf{f} \cdot (\mathbf{q} \times \mathbf{w} + \mathbf{v}) + (\mathbf{p} \times \mathbf{f} + \mathbf{t}) \cdot \mathbf{w})\, \delta t \\ &= (\mathbf{f} \cdot \mathbf{v} + \mathbf{t} \cdot \mathbf{w} - (\mathbf{p} - \mathbf{q}) \cdot (\mathbf{w} \times \mathbf{f}))\, \delta t = \mathscr{P}\, \delta t. \quad (9) \end{aligned}$$

The instantaneous quantity $\mathscr{P}$ is called the *infinitesimal work* by Murray et al.[27] and the "rate of work done" by Hunt[17] and Phillips.[22] We call this quantity the *virtual work* of W acting on T with the understanding it is associated with a virtual time period.

To simplify the computation of virtual work, we follow Kumar[5] and introduce the 6×6 matrix Π which interchanges the vector components of a screw, that is:

$$\Pi\mathsf{S} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{s} \\ \mathbf{u} \end{Bmatrix} = \begin{Bmatrix} \mathbf{u} \\ \mathbf{s} \end{Bmatrix}. \quad (10)$$

Lipkin and Duffy[29] describe (10) as a transformation of a screw from "ray coordinates" to "axial coordinates." For convenience, we introduce the notation $\check{\mathsf{T}} = \Pi\mathsf{T}$ to simplify the equations that follow. This notation allows us to write the virtual work in the form:

$$\mathscr{P} = \mathsf{W}^T \check{\mathsf{T}} = \mathbf{f} \cdot \mathbf{v} + \mathbf{t} \cdot \mathbf{w} - (\mathbf{p} - \mathbf{q}) \cdot (\mathbf{w} \times \mathbf{f}). \quad (11)$$

If the virtual work of a wrench acting on a twist is zero, then the two screws are said to be *reciprocal*.

3. THE JACOBIAN OF A PLATFORM MANIPULATOR

The Jacobian, $[J]$, of a single serial chain manipulator is given by:[12,27]

$$\check{\mathsf{T}} = \left[\check{\mathsf{S}}_1, \check{\mathsf{S}}_2, \ldots, \check{\mathsf{S}}_6\right] = [J]\dot{\bar{\theta}}, \quad (12)$$

where T is the twist of the end-effector and S_i are the partial twists associated with each joint. For a platform manipulator, we have the set of similar equations:

$$\check{\mathsf{T}} = \left[\check{\mathsf{S}}_1^{(i)}, \check{\mathsf{S}}_2^{(i)}, \ldots, \check{\mathsf{S}}_6^{(i)}\right] = [J^{(i)}]\dot{\bar{\theta}}^{(i)}, \qquad i = 1, \ldots, m, \quad (13)$$

where T, the twist of the platform, is the same for each of the m serial chains.

The Jacobian also relates joint torques, $\bar{\tau} = (\tau_1, \ldots, \tau_6)^T$, of a serial chain to the resultant wrench on the end-effector by the relation:

$$\mathsf{W} = [J]^T = \bar{\tau}. \quad (14)$$

Let the resultant wrench applied by the ith supporting chain of a platform manipulator be $\mathsf{W}^{(i)}$, then we have the joint torques $\bar{\tau}^{(i)} = (\tau_1^{(i)}, \ldots, \tau_6^{(i)})^T$, $i = 1, 2, \ldots, m$ given by

$$\begin{aligned} &\bar{\tau}^{(1)} = [J^{(1)T}]\mathsf{W}^{(1)}, \qquad \bar{\tau}^{(2)} = [J^{(2)T}]\mathsf{W}^{(2)}, \ldots, \\ &\qquad\qquad \bar{\tau}^{(m)} = [J^{(m)T}]\mathsf{W}^{(m)}. \quad (15) \end{aligned}$$

Invert each of these equations to determine the applied wrench $\mathsf{W}^{(i)}$ in terms of the applied joint torques, $\tau^{(i)}$, that is

$$\mathsf{W}^{(i)} = [J^{(i)T}]^{-1}\bar{\tau}^{(i)} = \left[\mathsf{F}_1^{(i)}, \mathsf{F}_2^{(i)}, \ldots, \mathsf{F}_6^{(i)}\right]\bar{\tau}^{(i)}. \quad (16)$$

The wrench $\mathsf{F}_j^{(i)}$ represents the contribution of the jth actuator of the ith chain, and is called an *actuator screw*.

The resultant force and torque applied to the platform is the sum

$$\begin{aligned} \mathsf{W} &= \sum_{i=1}^{m} \mathsf{W}^{(i)} \\ &= \left[[J^{(1)T}]^{-1}, [J^{(2)T}]^{-1}, \ldots, [J^{(m)T}]^{-1}\right] \begin{bmatrix} \bar{\tau}^{(1)} \\ \cdots \\ \bar{\tau}^{(m)} \end{bmatrix}. \quad (17) \end{aligned}$$

Substitute $[J^{(i)T}]^{-1}$ from (16) into this to obtain

$$\mathbf{W} = \Big[\mathbf{F}_1^{(1)}, \mathbf{F}_2^{(1)}, \ldots, \mathbf{F}_6^{(1)}; \mathbf{F}_1^{(2)}, \mathbf{F}_2^{(2)}, \ldots, \mathbf{F}_6^{(2)}; \ldots; \mathbf{F}_1^{(m)}, \mathbf{F}_2^{(m)}, \ldots, \mathbf{F}_6^{(m)}\Big] \begin{bmatrix} \bar{\tau}^{(1)} \\ \cdots \\ \bar{\tau}^{(6)} \end{bmatrix} = [\Gamma]\bar{\tau}. \tag{18}$$

The matrix $[\Gamma]$ is the Jacobian of the platform manipulator.[5] The transpose of $[\Gamma]$ defines the joint rates of the manipulator in terms of the desired twist.[30] See also Collins and Long.[10] This matrix also appears as the "grip matrix" in the study of grasping with a mechanical hand.[2,31,27]

The screw **W** depends on the configuration of all the serial chains of the in-parallel manipulator. Configurations for which the rank of $[\Gamma]$ is less than six are called *singular*.

4. CONDITIONS FOR LINE-BASED SINGULARITIES

We now focus our attention on platform manipulators that have a total of six actuators, and introduce conditions that ensure that the applied wrenches are pure forces. For these systems it is possible to characterize all singular configurations in terms of the geometry of linearly dependent sets of lines. We call these configurations *line-based singularities*. For serial chains consisting of hinges and sliders, we present a convenient means to determine the lines of action of these forces.

Each arm supporting the platform of the manipulator system is assumed to have the structure of a serial robotic arm that can provide six degree-of-freedom movement of the platform. We further assume that each arm has at least one joint at which a nonzero torque is applied, otherwise, it does not contribute to defining singular configurations. Thus, we are considering platforms supported by at most six serial chains.

Finally, we require each of the m serial chains to satisfy the following conditions:

- The last three joints are equivalent to a spherical joint at a point $\mathbf{c}_i$ of the platform—we consider this to be the point of attachment of the chain to the platform.
- The last three joints are unactuated, which means they do not apply any torque to the platform.

These conditions ensure that torque cannot be applied to the platform by a single chain. The result is that individual columns of the Jacobian $[\Gamma]$ cannot have pure moment terms, and, therefore, they must be Plücker vectors of lines. Systems that satisfy these conditions must have least three supporting serial chains to resist an arbitrary external force and torque; therefore, $3 \le m \le 6$.

These kinematic conditions are typical of many platform manipulator systems, particularly those based on the Stewart platform. Figure 3 shows the general structure and actuation schemes for platform manipulators that have purely line based singularities. Note that contact points on the platform and base can be generally located; and, it is possible to apply the actuator screws with one or more serial chains.

If, in addition to the above conditions, we require the actuated joints in each serial chain provide purely rotary or linear movement, known as *revolute* and *prismatic* joints, then each actuator screw, $\mathbf{F}_i$, $i = 1, 2, 3$, is uniquely defined by the location of the axis of its associated joint. This often makes it possible to determine the line of action of these forces by inspection.

To show this, we first derive a fundamental relationship between the actuator screws $\mathbf{F}_j$ and joint screws $\mathbf{S}_k$ of a general six degree-of-freedom serial chain. Using the identity $[J^T]^{-1} = [J^{-1}]^T$ and (16),

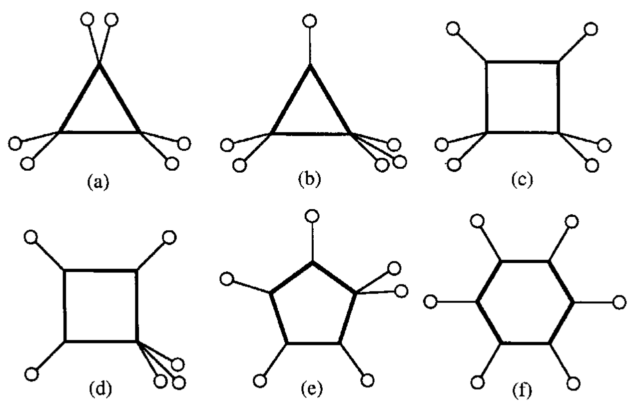

Figure 3. The six basic designs of spatial platform manipulators with actuation schemes that admit line-based singularities. The circles denote the actuator screws of the supporting chains acting on the platform.

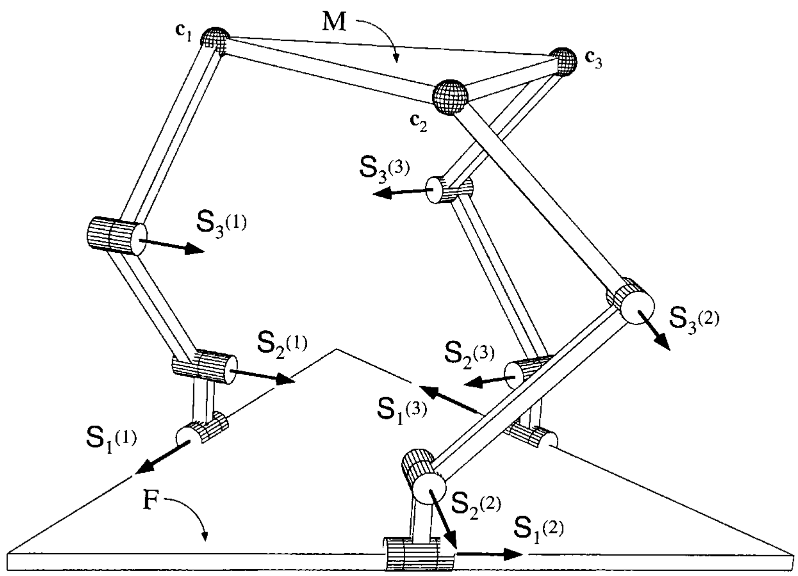

Figure 4. An example of a platform manipulator with line-based singularities.

we have

$$[J^{-1}] = \begin{bmatrix} \mathbf{F}_1^T \\ \cdots \\ \mathbf{F}_6^T \end{bmatrix}. \qquad (19)$$

Now compute:

$$[J^{-1}][J] = \begin{bmatrix} \mathbf{F}_1^T \\ \cdots \\ \mathbf{F}_6^T \end{bmatrix} \left[\hat{\mathbf{S}}_1, \check{\mathbf{S}}_2, \ldots, \check{\mathbf{S}}_6\right] = [I]. \qquad (20)$$

The conclusion is that *each the actuator screw* $\mathbf{F}_j$ *generates zero virtual work when it acts on the joint screw* $\mathbf{S}_k$ *for* $j \neq k$, *that is, these screws are reciprocal.*

The Jacobian of a serial chain, in which the first three joints are revolute or prismatic joints and the last three are equivalent to a spherical joint, has columns that are Plücker vectors representing each of the joint axes. By (20), the line of action of $\mathbf{F}_1$ must be reciprocal to the axes $\mathbf{S}_2, \ldots, \mathbf{S}_6$, which means it intersects these five lines. To determine the line $\mathbf{F}_1$, consider the plane defined by the point of attachment **c** and the joint axis $\mathbf{S}_3$. The axis $\mathbf{S}_2$ intersects this plane in a point P. The line joining **c** and P is uniquely determined, and must be the line of action of $\mathbf{F}_1$.[7] It may happen that $\mathbf{S}_2$ is parallel to the plane defined by $\mathbf{S}_3$ and **c**. In this case $\mathbf{F}_1$ is the line in the plane through **c** parallel to $\mathbf{S}_2$, which is said to intersect $\mathbf{S}_2$ "at infinity." This construction yields an equivalent line for each actuator of the chain.

See Figure 4 for an example of a platform manipulator that has line based singularities; notice it may be either the basic design "a" or "b" in Figure 3, depending on which joints are actuated.

5. CLASSIFICATION OF LINE-BASED SINGULARITIES

A platform manipulator with six actuated joints has a Jacobian of the form:

$$\mathbf{W} = [\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_6] \begin{bmatrix} \tau_1 \\ \cdots \\ \tau_6 \end{bmatrix} = [\Gamma]\bar{\tau}, \qquad (21)$$

where the actuated joints are now numbered $i = 1, \ldots, 6$. A singular configuration of this manipulator system is identified by linear dependence among the six-dimensional vectors $\mathbf{F}_i$.

If the platform manipulator meets the conditions described in the previous section then each of the actuator screws, $\mathbf{F}_i$, is the Plücker vector of a line and has the form $\mathbf{F}_i = (\mathbf{f}_i, \mathbf{p}_i \times \mathbf{f}_i)$. It is easy to see that these screws have the property that they are reciprocal to themselves, that is $\mathbf{F}_i^T \check{\mathbf{F}}_i = 0$.

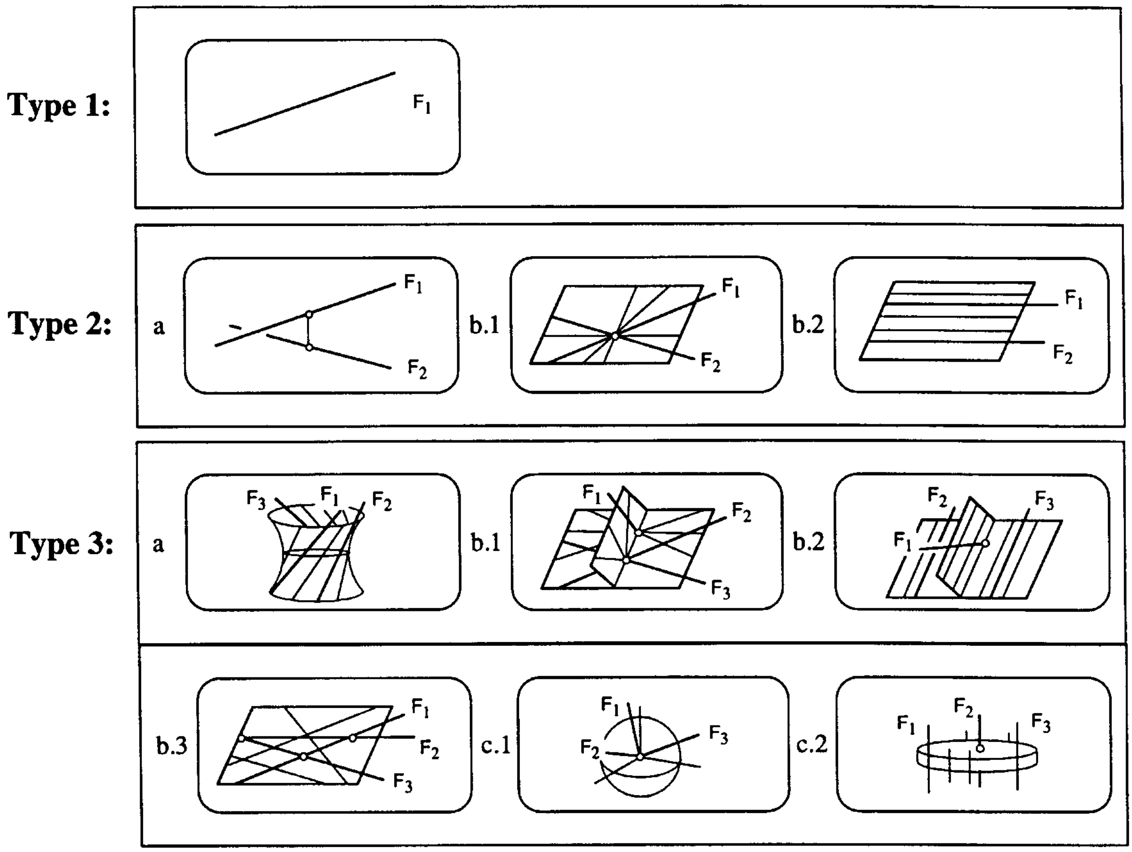

Figure 5. A classification of the sets of lines that are linearly dependent on one, two, or three given lines.

It is useful at this point to note that, if two lines $\mathbf{L} = (\mathbf{s}, \mathbf{r} \times \mathbf{s})$ and $\mathbf{F} = (\mathbf{f}, \mathbf{p} \times \mathbf{f})$ are reciprocal, then they must either intersect or be parallel. To see this let $\mathbf{N}$ be the common normal to $\mathbf{L}$ and $\mathbf{F}$, and $\mathbf{d}$ and $\mathbf{c}$ be their points of intersection, respectively, with $\mathbf{N}$. Then we have:

$$\mathbf{F}^T\check{\mathbf{L}} = (\mathbf{p} - \mathbf{r}) \cdot \mathbf{s} \times \mathbf{f} = (\mathbf{c} - \mathbf{d}) \cdot \mathbf{s} \times \mathbf{f} = -d \sin \alpha |\mathbf{f}|\, |\mathbf{s}|, \tag{22}$$

where d is the distance between the two lines along $\mathbf{N}$ and α is the angle about $\mathbf{N}$ measured from $\mathbf{L}$ to $\mathbf{F}$. Clearly, if the two lines intersect then $d = 0$ and the lines are reciprocal. Similarly, this is true if the lines are parallel such that $\sin \alpha = 0$. In this latter case, the two lines are said to "intersect at infinity." Thus, $\mathbf{F}^T\check{\mathbf{L}} = 0$ is described simply as the condition that the two lines intersect.

A platform manipulator achieves a singular configuration when any one of the actuator screws becomes linearly dependent on the others. We say the singularity is of *type n* if the line defined by the actuator screw $\mathbf{F}$ is dependent on no less than n other actuator screws. In what follows, we provide a complete description of linearly dependent sets of lines available to platform manipulators. Merlet's notation is used for the general classes, though, we include additional subcases. Figures 5 and 6 provide illustrations of these various distributions of lines.

5.1. Type 1 Singularities

If any one of the six actuator screws, $\mathbf{F}$, in a platform manipulator is linearly dependent on one of the remaining five, denoted $\mathbf{F}_1$, then the system is in a type 1 singularity. This is equivalent to the condition that:

$$\mathcal{T}_1\colon \mathbf{F} = k_1 \mathbf{F}_1, \tag{23}$$

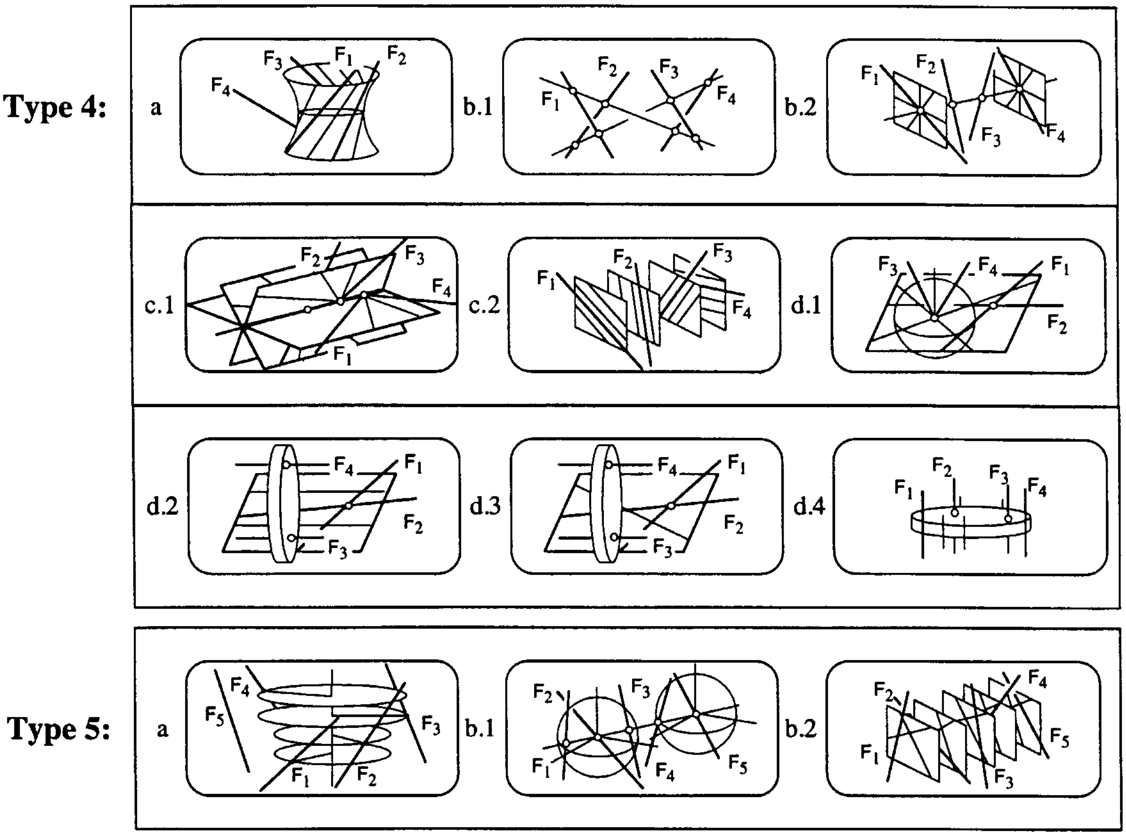

Figure 6. A classification of the sets of lines linearly dependent on four or five given lines.

for some scalar k_1. It is easy to see that for this to occur $\mathbf{F}$ and $\mathbf{F}_1$ must define the same line.

5.2. Type 2 Singularities

A type 2 singularity occurs when one of the actuator screws, $\mathbf{F}$, is linearly dependent on two others, denoted $\mathbf{F}_1$ and $\mathbf{F}_2$, and not on either one independently. This means nonzero scalars, k_1 and k_2, exist such that:

$$\mathcal{T}_2\colon \mathbf{F} = k_1\mathbf{F}_1 + k_2\mathbf{F}_2. \tag{24}$$

In general, this equation defines a set of screws known as a *two-system*.[17] However, we are concerned only with actuator screws $\mathbf{F}$ that are Plücker vectors of a line. Thus, we have the additional requirement:

$$\mathbf{F}^T\check{\mathbf{F}} = k_1^2(\mathbf{F}_1^T\check{\mathbf{F}}_1) + 2k_1k_2(\mathbf{F}_1^T\check{\mathbf{F}}_2) + k_2^2(\mathbf{F}_2^T\check{\mathbf{F}}_2) = 0. \tag{25}$$

Since $\mathbf{F}_1$ and $\mathbf{F}_2$ are lines, the terms $\mathbf{F}_i^T\check{\mathbf{F}}_i$ are zero, and we find that $\mathbf{F}$ cannot be a line unless $\mathbf{F}_1^T\check{\mathbf{F}}_2 = 0$. This means the two lines $\mathbf{F}_1$ and $\mathbf{F}_2$ must intersect; see Eq. (22). When this is true, the lines of $\mathcal{T}_2$ lie in the plane defined by $\mathbf{F}_1$ and $\mathbf{F}_2$ and pass through their point of intersection. This is known as a *pencil* of lines.

5.2.1. Type 2a

Merlet[6] and Dandurand[7] include two skew lines as a case of a linearly dependent set of lines. However, two skew lines cannot generate a third line which is linearly dependent on both of them, so this is not, strictly speaking, a type 2 singularity. A third line must actually coincide with one of the two lines, as in the type 1 singularity.

5.2.2. Type 2b

The type 2 singularity occurs when $\mathbf{F}_1$ and $\mathbf{F}_2$ intersect in a point $\mathbf{p}$. In this case the two lines define a plane and the pencil of lines through $\mathbf{p}$ in this plane are the lines of $\mathscr{T}_2$. There are two cases depending on whether or not $\mathbf{p}$ is a finite point.

1. If the two lines intersect in a finite point, which means F_1 and F_2 are not parallel, then the lines of $\mathscr{T}_2$ form a planar pencil through their point of intersection.
2. If F_1 and F_2 are parallel then they are said to intersect "at infinity." In this case, $\mathscr{T}_2$ is a set of parallel lines in the plane containing F_1 and F_2.

5.3. Type 3 Singularities

When one actuator screw F is dependent on three other actuators screws, F_1, F_2, and F_3 the platform manipulator system is in a type 3 singularity. This occurs when k_i, $i=1,2,3$ exist so F is given by

$$\mathscr{T}_3\colon \mathsf{F} = k_1\mathsf{F}_1 + k_2\mathsf{F}_2 + k_3\mathsf{F}_3. \tag{26}$$

The requirement that F also be a line yields the relation:

$$\mathsf{F}^T\check{\mathsf{F}} = 2k_1k_2\left(\mathsf{F}_1^T\check{\mathsf{F}}_2\right) + 2k_1k_3\left(\mathsf{F}_1^T\check{\mathsf{F}}_3\right) = 2k_2k_3\left(\mathsf{F}_2^T\check{\mathsf{F}}_3\right) = 0. \tag{27}$$

Notice that we have dropped the terms $\mathsf{F}_i^T\check{\mathsf{F}}_i = 0$. Equation (27) is easily solved to obtain:

$$k_3 = \frac{-k_1k_2\left(\mathsf{F}_1^T\check{\mathsf{F}}_2\right)}{k_1\left(\mathsf{F}_1^T\check{\mathsf{F}}_3\right) + k_2\left(\mathsf{F}_2^T\check{\mathsf{F}}_3\right)}. \tag{28}$$

This result combines with (26) to define $\mathscr{T}_3$ as a one dimensional set of lines which is known to be a quadric surface, $\mathscr{Q}$.

Another view of this quadric, $\mathscr{Q}$, is obtained by considering the lines $\mathsf{L} = (\mathbf{s}, \mathbf{r}\times\mathbf{s})$ that intersect F_1, F_2, and F_3, defined by the matrix equation:

$$[\mathsf{F}_1, \mathsf{F}_2, \mathsf{F}_3]\check{\mathsf{L}} = 0. \tag{29}$$

Any line L satisfying this equation intersects all the lines of $\mathscr{T}_3$ that define $\mathscr{Q}$, and therefore must lie on $\mathscr{Q}$. This provides a convenient way to derive its algebraic expression in terms of the point coordinates $\mathbf{r} = (x, y, z)^T$. Given the lines $\mathsf{F}_i = (\mathbf{f}_i, \mathbf{p}_i\times\mathbf{f}_i)$, we manipulate (29) to obtain:

$$\begin{bmatrix} ((\mathbf{p}_1-\mathbf{r})\times\mathbf{f}_1)^T \\ ((\mathbf{p}_2-\mathbf{r})\times\mathbf{f}_2)^T \\ ((\mathbf{p}_3-\mathbf{r})\times\mathbf{f}_3)^T \end{bmatrix} \begin{Bmatrix} s_x \\ s_y \\ s_z \end{Bmatrix} = 0. \tag{30}$$

Clearly, there is a solution for $\mathbf{s} = (s_x, s_y, s_z)^T$ only if the determinant of the coefficient matrix is zero. This determinant can be expressed as the triple product:

$$\mathscr{Q}\colon ((\mathbf{p}_1-\mathbf{r})\times\mathbf{f}_1) \cdot[((\mathbf{p}_2-\mathbf{r})\times\mathbf{f}_2)\times((\mathbf{p}_3-\mathbf{r})\times\mathbf{f}_3))] = 0. \tag{31}$$

The cubic terms cancel because $(\mathbf{r}\times\mathbf{f}_1)\cdot((\mathbf{r}\times\mathbf{f}_2)\times(\mathbf{r}\times\mathbf{f}_3)) = 0$, and (31) is the equation of the quadric surface $\mathscr{Q}$. The lines L and F define the two separate sets of rulings on this quadric, known as *reguli*.

5.3.1. Type 3a

In general, the set of lines $\mathscr{T}_3$ is the regulus of lines F defined by (28) lying on the quadric $\mathscr{Q}$ defined by (31).

5.3.2. Type 3b

If any two of the three lines, F_1, F_2, and F_3, intersect, then the quadric $\mathscr{Q}$ degenerates to a pair of planes. To see this, let $\mathbf{p} = \mathbf{p}_2 = \mathbf{p}_3$, then (1) takes the form:

$$((\mathbf{r}-\mathbf{p})\cdot(\mathbf{p}-\mathbf{p}_1)\times\mathbf{f}_1)((\mathbf{r}-\mathbf{p})\cdot\mathbf{f}_2\times\mathbf{f}_3) = 0, \tag{32}$$

which is the product of two linear equations in the coordinates of $\mathbf{r} = (x, y, z)^T$. These equations define the two planes,

$$\mathscr{P}_1\colon (\mathbf{r}-\mathbf{p})\cdot(\mathbf{p}-\mathbf{p}_1)\times\mathbf{f}_1 = 0,$$
$$\mathscr{P}_2\colon (\mathbf{r}-\mathbf{p})\cdot\mathbf{f}_2\times\mathbf{f}_3 = 0. \tag{33}$$

We can distinguish three cases:

1. If the planes are distinct and not parallel, then their intersection is a finite line, denoted L. This line must pass through the point of intersection $\mathbf{p}$ of F_2 and F_3 and intersect F_1 in a point $\mathbf{a}$. The set $\mathscr{T}_3$ is all the lines through $\mathbf{a}$ in $\mathscr{P}_1$ and all those through $\mathbf{p}$ in $\mathscr{P}_2$. This case includes the situation when two of the lines, say F_1 and F_2, both intersect the third line, F_3. The two planar pencils are defined by the pairs of lines F_1 and F_3, and F_2 and F_3.

2. If the planes are distinct and parallel, then their intersection $\mathbf{L}$ is at infinity. This occurs when two of the three lines are parallel. Thus, $\mathscr{T}_3$ consists of two planes of parallel lines.
3. If the three lines lie in one plane, then the quadric degenerates to two coincident planes. In this case, $\mathscr{T}_3$ consists of all lines in that plane.

5.3.3. Type 3c

If the three lines, $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, intersect in the same point $\mathbf{p} = \mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}_3$, then the quadric generates to a bundle of lines through $\mathbf{p}$. There are two cases depending on whether this point is finite, or at infinity.

1. If the point of intersection $\mathbf{p}$ is not at infinity, then equation $\mathscr{T}_3$ is the set of all lines through $\mathbf{p}$.
2. If the three lines are parallel, then the point of intersection $\mathbf{p}$ is at infinity, and all lines in space parallel to these three lines form $\mathscr{T}_3$.

5.4. Type 4 Singularities

The type 4 singularities occur when one actuator screw $\mathbf{F}$ is linearly dependent on no fewer than four other actuator screws, that is:

$$\mathscr{T}_4\colon \mathbf{F} = k_1\mathbf{F}_1 + k_2\mathbf{F}_2 + k_3\mathbf{F}_3 + k_4\mathbf{F}_4. \tag{34}$$

This is a two dimensional set of lines known as a *linear congruence*.

We analyze this set of lines by considering the screws $\mathbf{S} = (\mathbf{s}, \mathbf{u})$ that are reciprocal to all four lines $\mathbf{F}_i$, $i = 1, 2, 3, 4$, and therefore are reciprocal to the entire set $\mathscr{T}_4$. The screws $\mathbf{S}$ must satisfy the four linear condition $\mathbf{S}^T\check{\mathbf{F}}_i = 0$, $i = 1, 2, 3, 4$, which we write in matrix form as:

$$\left[\check{\mathbf{F}}_1, \check{\mathbf{F}}_2, \check{\mathbf{F}}_3, \check{\mathbf{F}}_4\right]^T \mathbf{S} = 0. \tag{35}$$

Let $[A]$ be the 4×4 submatrix formed from the first four columns of the 4×6 coefficient matrix in (35), and let $\mathbf{b}_1$ and $\mathbf{b}_2$ be the fifth and sixth columns, so we have $[\check{\mathbf{F}}_1, \check{\mathbf{F}}_2, \check{\mathbf{F}}_3, \check{\mathbf{F}}_4]^T = [A, \mathbf{b}_1, \mathbf{b}_2]$. Equation (35) can now be solved to determine:

$$\mathbf{S}_1 = \begin{Bmatrix} -[A^{-1}]\mathbf{b}_1 \\ 1 \\ 0 \end{Bmatrix} \quad \text{and} \quad \mathbf{S}_2 = \begin{Bmatrix} -[A^{-1}]\mathbf{b}_2 \\ 0 \\ 1 \end{Bmatrix}. \tag{36}$$

Actually, any linear combination of these two screws, $\mathbf{L} = s\mathbf{S}_1 + t\mathbf{S}_2$, will satisfy (35). This is known as *two-system* of screws.

We now consider whether or not the two-system spanned by $\mathbf{S}_1$ and $\mathbf{S}_2$ contains any lines. This is determined by the roots of the quadratic equation:

$$\mathbf{L}^T\check{\mathbf{L}} = s^2\left(\mathbf{S}_1^T\check{\mathbf{S}}_1\right) + 2st\left(\mathbf{S}_1^T\check{\mathbf{S}}_2\right) + t^2\left(\mathbf{S}_2^T\check{\mathbf{S}}_2\right) = 0. \tag{37}$$

If the roots are complex then there are no real lines in the two-system, in which case $\mathscr{T}_4$ is called an *elliptic* linear congruence. If there are two real roots then, two lines exist that intersect every line $\mathscr{T}_4$, which is termed a *hyperbolic* linear congruence. Finally, if (37) yields a double root then $\mathscr{T}_4$ is a *parabolic* linear congruence.

5.4.1. Type 4a

If the two roots of (37) are imaginary, then the two-system generated by $\mathbf{S}_1$ and $\mathbf{S}_2$ contains screws but no lines. In this case, the set of lines, $\mathscr{T}_4$, is an *elliptic linear congruence*. Hunt[17] shows that these lines form concentric hyperboloids about the common normal to the axes of the two screws $\mathbf{S}_1$ and $\mathbf{S}_2$. The relationship between these screws and the distribution of the lines of $\mathscr{T}_4$ deserves further study.

5.4.2. Type 4b

If the two roots of (37) are real and distinct, then two lines, $\mathbf{L}_a$ and $\mathbf{L}_b$, exist that intersect the entire set of lines, $\mathscr{T}_4$, termed a *hyperbolic congruence*. Assume these two lines are skew, then we have two cases depending on whether or not one of these lines lies at infinity. A line at infinity has the Plücker vector of the form $\mathbf{L} = (0, \mathbf{v})$; this is also described as a screw with "infinite pitch."

1. If the two lines $\mathbf{L}_a$ and $\mathbf{L}_b$ are finite and skew, then $\mathscr{T}_4$ is the set of lines that intersect these two lines.
2. If one of the lines, say $\mathbf{L}_b$ is at infinity, then it has the form $\mathbf{L}_b = (0, \mathbf{v})$. In this case, $\mathbf{L}_a$ intersects the set of parallel planes orthogonal to $\mathbf{v}$, through $\mathbf{L}_b$, in a series of points. The congruence $\mathscr{T}_4$ consists of the pencil of lines through this point on each parallel plane.

The cases in which the two lines $\mathbf{L}_a$ and $\mathbf{L}_b$ intersect each other are traditionally identified separately as "degenerate." We consider these cases in Type 4d below.

5.4.3. Type 4c

If Eq. (37) has a double root, then we have a double line $\mathbf{L} = \mathbf{L}_a = \mathbf{L}_b$. In this case, the set $\mathcal{T}_4$ is known as a *parabolic linear congruence*. We can distinguish two cases, depending on whether or not this double line is at infinity.

1. Each point on the finite line $\mathbf{L}$ is the vertex of a planar pencil of lines. Each pencil lies in a different plane passing through the $\mathbf{L}$.
2. If the two lines coincide at infinity, that is $\mathbf{L}_a = \mathbf{L}_b = (0, \mathbf{v})$, then all lines in planes orthogonal to $\mathbf{v}$ form the congruence, $\mathcal{T}_4$.

5.4.4. Type 4d

The situations in which the two lines of the hyperbolic congruence intersect are termed *degenerate*. We identify four subcases depending on whether one or both of the lines are finite or at infinity.

1. If the two lines defined by (37) intersect, then $\mathcal{T}_4$ consists of all the lines in the plane defined by the two lines $\mathbf{L}_a$ nd $\mathbf{L}_b$, and all lines in space that pass through their point of intersection.
2. If the lines $\mathbf{L}_a$ and $\mathbf{L}_b$ are parallel, which means they intersect at a point at infinity, then $\mathcal{T}_4$ consists of all lines in the plane containing these two lines, and all lines in space parallel to them.
3. If one line $\mathbf{L}_a$ is finite and the other line $\mathbf{L}_b = (0, \mathbf{v})$ is at infinity, then $\mathbf{L}_a$ is contained entirely in one of the parallel planes orthogonal to $\mathbf{v}$. In this case the congruence consists of the lines parallel to $\mathbf{L}_a$, together with all lines in the plane that contains $\mathbf{L}_a$.
4. If both lines lie at infinity, then we have $\mathbf{L}_a = (0, \mathbf{v}_a)$ and $\mathbf{L}_b = (0, \mathbf{v}_b)$, in which case the congruence is formed by all lines in the direction $\mathbf{s} = \mathbf{v}_a \times \mathbf{v}_b$. For this to occur the four lines $\mathbf{F}_i$, $i = 1, 2, 3, 4$ must be parallel, which means they form a type 3c.2 singular configuration; thus we conclude that this case cannot occur independently.

5.5. Type 5 Singularities

The final singularity that we consider occurs when an actuator screw is a linear combination of the remaining five actuator screws. In this case, we have:

$$\mathcal{T}_5 : \mathbf{F} = k_1\mathbf{F}_1 + k_2\mathbf{F}_2 + k_3\mathbf{F}_3 + k_4\mathbf{F}_4 + k_5\mathbf{F}_5. \quad (38)$$

Let the six minors of the 6×5 matrix $[F] = [\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_5]$ be denoted M_i—the ith minor is the determinant of the 5×5 matrix obtained by removing row i from $[F]$. Thus, the set of lines $\mathcal{T}_5$ satisfy the linear equation:

$$M_1 f_1 + M_2 f_2 + M_3 f_3 + M_4 f_4 + M_5 f_5 + M_6 f_6 = 0. \quad (39)$$

This is the equation of a *linear complex*. Assemble the six coefficients M_i into the screw $\mathbf{M} = (M_4, M_5, M_6, M_1, M_2, M_3)^T$ with axis $\mathbf{L} = (\mathbf{s}, \mathbf{r} \times \mathbf{s})$ and pitch μ; that is, $\mathbf{M} = (\mathbf{s}, \mathbf{r} \times \mathbf{s} + \mu\mathbf{s})$. Then, (39) can be written in terms of $\mathbf{M}$ and $\mathbf{F} = (\mathbf{f}, \mathbf{p} \times \mathbf{f})$, to obtain:

$$\mathbf{M}^T \check{\mathbf{F}} = \mathbf{L}^T \check{\mathbf{F}} + \mu\mathbf{s} \cdot \mathbf{f} = 0. \quad (40)$$

Let the distance from $\mathbf{L}$ to $\mathbf{F}$ along their common normal be d and the angle about the common normal be α, then (40) reduced to:

$$\mathbf{M}^T \check{\mathbf{F}} = -d \sin\alpha + \mu \cos\alpha = 0 \quad \text{or} \quad d \tan\alpha = \mu. \quad (41)$$

This equation shows that lines of this complex that are a distance d from the axis $\mathbf{L}$ lie at the angle $\alpha = \arctan(\mu/d)$ about the common normal.

5.5.1. Type 5a

In general, the complex $\mathcal{T}_5$ is the set of lines tangent to helices with the line $\mathbf{L}$, described above, as its axis. If the radius of the helix is d then its lead is $2\pi d^2/\mu$.[15]

5.5.2. Type 5b

If the components of the screw $\mathbf{M}$ are such that they form the Plücker coordinates of a line, that is $\mathbf{M}^T \check{\mathbf{M}} = 0$, then $\mathcal{T}_5$ is a *special linear complex*. There are two cases depending on whether or not $\mathbf{M}$ is at infinity.

1. If $\mathbf{M}$ is a real line, then $\mathcal{T}_5$ consists of all lines in space that intersect $\mathbf{M}$.
2. If $\mathbf{M}$ has the form $\mathbf{M} = (0, \mathbf{v})$, which means it lies at infinity, then $\mathcal{T}_5$ consists of all lines in the set of planes orthogonal to $\mathbf{v}$. This means that the common normal to every pair of lines in the complex is parallel to the direction $\mathbf{v}$.

6. DISCUSSION

Singularities occur in platform manipulators either as a result of a singular configuration within a supporting chain, or due to interdependence of actuator screws on separate chains. For the systems we are considering the serial chain singularities can only be of type 1 or 2, that is either two actuator screws fall on the same line, or three lie on the same plane.

If we assume no supporting chain is in a singular configuration, then we can determine the following examples of singular configurations in the basic platform designs.

1. A type 1 singularity occurs when an actuator screw at an attachment point $\mathbf{c}_i$ aligns with one at $\mathbf{c}_j$. This is equivalent to having the axes of two actuator screws coincide with the segment $\overline{\mathbf{c}_i\mathbf{c}_j}$.
2. A type 2b.1 singularity becomes possible in designs a, b, c, and e as follows. Let $\mathbf{c}_i$ be the point supported by two actuator screws, then if any other point $\mathbf{c}_j$ lies in the plane defined by these two lines and the associated actuator screw falls on the line $\overline{\mathbf{c}_i\mathbf{c}_j}$, then the system is singular.
3. The type 3c.1 singularity is found in designs b and d. Let $\mathbf{c}_i$ be the point supported by three actuator screws, then the singularity occurs when an actuator screw at $\mathbf{c}_j$ falls on the line $\overline{\mathbf{c}_i\mathbf{c}_j}$.
4. The degenerate hyperbolic congruence of type 4d.1 occurs in design d. Let the point supported by three actuators be $\mathbf{c}_i$, then the singularity occurs when actuator screws at two other points $\mathbf{c}_j$ and $\mathbf{c}_k$ fall in the plane defined by the triangle $\Delta\mathbf{c}_i\mathbf{c}_j\mathbf{c}_k$. This is an example of five actuator screws intersecting two lines, which are two edges of the platform.
5. The type 5b.1 singularity occurs in designs a and b when the plane defined by the screws of a two actuator chain coincides with the platform itself.

7. CONCLUSION

This paper defines the geometric characteristics of the structure of a platform manipulator that ensure that each column of its Jacobian is Plücker vector of a line. Singular configurations for these manipulator systems can be associated directly with linear combinations of lines which we call "line-based singularities." A singularity of *type n* occurs when one actuator screw is a linear combination of at least n other actuator screws. The loci associated with linear combinations of 1–5 lines are described to facilitate insight to the geometric characteristics of these singularities.

The insight and advice of Curtis Collins (UCI) and Gregory Long (Reflex Technologies) are gratefully acknowledged, as well as the support of National Science Foundation Grant DMII 9321936.

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